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TURÁN'S PROBLEM FOR TREES

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Abstract

For a forbidden graph L, let $ex(p;L)$ denote the maximal number of edges in a simple graph of order p not containing L. Let T_n denote the unique tree on n vertices with maximal degree $n-2$, and let $T_n^* = (V,E)$ be the tree on n vertices with $V = \{v_0, v_1, \dots, v_{n-1}\}$ and $E = \{v_0v_1, \dots, v_0v_{n-3}, v_{n-3}v_{n-2}, v_{n-2}v_{n-1}\}$. In the paper we give exact values of $ex(p;T_n)$ and $ex(p;T_n^*)$.

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1. Introduction

In the paper, all graphs are simple graphs. For a graph $G = (V(G), E(G))$ let $e(G) = |E(G)|$ be the number of edges in G and let $\Delta(G)$ be the maximal degree of G. For a family of forbidden graphs L, let $ex(p;L)$ denote the maximal number of edges in a graph of order p not containing any graphs in L. The corresponding Turán's problem is to evaluate $ex(p;L)$. For a graph G of order p, if G does not contain any graphs in L and $e(G) = ex(p;L)$, we say that G is an extremal graph. In the paper we also use $Ex(p;L)$ to denote the set of extremal graphs of order p not containing any graphs in L.

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Let $\mathbb{N}$ be the set of positive integers. Let $p,n\in\mathbb{N}$ with $p\ge n\ge 2$. For a given tree T on n vertices, it is difficult to determine the value of $ex(p;T)$. The famous Erdös-Sós conjecture asserts that $ex(p;T)\le \frac{(n-2)p}{2}$. For the progress on the Erdös-Sós conjecture, see [2,6,7,8]. Write $p=k(n-1)+r$, where $k\in\mathbb{N}$ and $r\in\{0,1,\dots,n-2\}$. Let P_n be the path on n vertices. In [3] Faudree and Schelp showed that

$$ex(p;P_n)=k\binom{n-1}{2}+\binom{r}{2}. \tag{1.1}$$

In the special case $r=0$, (1.1) is due to Erdös and Gallai [1]. Let $K_{1,n-1}$ denote the unique tree on n vertices with $\Delta(K_{1,n-1})=n-1$, and let T_n denote the unique tree on n vertices with $\Delta(T_n)=n-2$. In Section 2 we determine $ex(p;K_{1,n-1})$, and in Section 3 we obtain the exact value of $ex(p;T_n)$.

For $n\ge 4$ let $T_n^*=(V,E)$ be the tree on n vertices with $V=\{v_0,v_1,\dots,v_{n-1}\}$ and $E=\{v_0v_1,\dots,v_0v_{n-3},v_{n-3}v_{n-2},v_{n-2}v_{n-1}\}$. In Section 4 we completely determine the value of $ex(p;T_n^*)$.

In addition to the above notation, throughout the paper we also use the following notation: $[x]$—the greatest integer not exceeding x, $d(v)$—the degree of the vertex v in a graph, $\Gamma(v)$—the set of vertices adjacent to the vertex v, $d(u,v)$—the distance between the two vertices u and v in a graph, K_n—the complete graph on n vertices, $K_{m,n}$—the complete bipartite graph with m and n vertices in the bipartition, $G[V_0]$—the subgraph of G induced by vertices in the set V_0, $G-V_0$—the subgraph of G obtained by deleting vertices in V_0 and all edges incident with them, $G-M$—the graph obtained by deleting all edges in M from the graph G, $G+M$—the graph obtained by adding all edges in M from the graph G.

2. The Evaluation of $ex(p;K_{1,n-1})$

Theorem 2.1. *Let $p,n\in\mathbb{N}$ with $p\ge n-1\ge 1$. Then $ex(p;K_{1,n-1})=[\frac{(n-2)p}{2}]$.*

Proof. Clearly $ex(n-1;K_{1,n-1})=e(K_{n-1})=\frac{(n-1)(n-2)}{2}$. Thus the result is true for $p=n-1$. Now we assume $p\ge n$. Suppose that G is a graph of order p without $K_{1,n-1}$. Then clearly $\Delta(G)\le n-2$ and so $2e(G)=\sum_{v\in V(G)}d(v)\le p\Delta(G)\le (n-2)p$. Hence, $ex(p;K_{1,n-1})\le\frac{(n-2)p}{2}$. As $ex(p;K_{1,n-1})$ is an integer, we have

$$ex(p;K_{1,n-1})\le\Big[\frac{(n-2)p}{2}\Big]. \tag{2.1}$$

Clearly $ex(p;K_{1,1})=0$. So the result holds for $n=2$. As $[\frac{p}{2}]K_2$ does not contain $K_{1,2}$, we have $ex(p;K_{1,2})\ge[\frac{p}{2}]$. This together with (2.1) gives $ex(p;K_{1,2})=\big[\frac{p}{2}\big]$. So the result is true for $n=3$.

Suppose that G is a Hamilton cycle with p vertices. Then G does not contain $K_{1,3}$. Thus we have $ex(p;K_{1,3}) \ge p$. Combining this with (2.1) yields $ex(p;K_{1,3}) = p$. So the result is true for $n=4$.

Now we assume $n \ge 5$. By (2.1), it suffices to show that $ex(p;K_{1,n-1}) \ge [\frac{(n-2)p}{2}]$. Set $k=[\frac{p+1}{2}]$, $V=\{1,2,\dots,2k\}$ and $M=\{12,34,\cdots,(2k-1)(2k)\}$. Let us consider the following four cases.

Case 1. $2 \mid p$ and $2 \nmid n$. Set $G=(V,E)$, where

$$E=\big\{ij \mid i,j\in V,\ j-i\in\{1,2k-1,k,k\pm 1,\dots,k\pm (n-5)/2\}\big\}.$$

Clearly G is an $(n-2)$-regular graph of order p and so G does not contain $K_{1,n-1}$. Hence, $ex(p;K_{1,n-1}) \ge e(G) = \frac{(n-2)p}{2} = [\frac{(n-2)p}{2}]$.

Case 2. $2 \mid p$ and $2 \mid n$. Set

$$E_1=\big\{ij \mid i,j\in V,\ j-i\in\{1,2k-1,k,k\pm 1,\dots,k\pm (n-4)/2\}\big\}.$$

Then $M\subset E_1$. Let $G=(V,E_1-M)$. We see that G is an $(n-2)$-regular graph of order p and so G does not contain $K_{1,n-1}$. Hence, $ex(p;K_{1,n-1}) \ge e(G) = \frac{(n-2)p}{2} = [\frac{(n-2)p}{2}]$.

Case 3. $2 \nmid p$ and $2 \mid n$. Let G be the $(n-2)$-regular graph of order $2k$ constructed in Case 2. Let

$$v_1=k-\frac{n}{2}+3,\ v_2=k-\frac{n}{2}+4,\dots,v_{n-3}=k+\frac{n}{2}-1 \quad\text{and}\quad v_{n-2}=2k.$$

Then clearly $v_1,\dots,v_{n-2}$ are all the vertices adjacent to the vertex 1. If $2 \mid k-\frac{n}{2}$, then $v_1,v_3,\dots,v_{n-5}$ are odd and so $v_1v_2,v_3v_4,\dots,v_{n-5}v_{n-4}\in M$. Thus, $v_1v_2,v_3v_4,\dots,v_{n-5}v_{n-4}\notin E(G)$. As $2k-(k+\frac{n}{2}-1)=k-\frac{n-2}{2}$, we see that $v_{n-3}v_{n-2}\notin E_1$ and so $v_{n-3}v_{n-2}\notin E(G)$. Let

$$G'=G-\{1\}+\{v_1v_2,v_3v_4,\dots,v_{n-5}v_{n-4},v_{n-3}v_{n-2}\}.$$

We see that G' is an $(n-2)$-regular graph of order p. Hence, $ex(p;K_{1,n-1}) \ge e(G') = \frac{(n-2)p}{2} = [\frac{(n-2)p}{2}]$.

If $2 \nmid k-\frac{n}{2}$, then $v_2,v_4,\dots,v_{n-4}$ are odd and so $v_2v_3,v_4v_5,\dots,v_{n-4}v_{n-3}\in M$. Thus, $v_2v_3,v_4v_5,\dots,v_{n-4}v_{n-3}\notin E(G)$. As $p+1=2k>n$ we have $k-\frac{n}{2}+3>3$ and so $2,3\notin\{v_1,\dots,v_{n-2}\}$. Clearly $2v_{n-2},3v_1\notin E_1$ and so $2v_{n-2},3v_1\notin E(G)$. Let

$$G'=G-\{1\}-\{23\}+\{v_2v_3,v_4v_5,\dots,v_{n-4}v_{n-3},3v_1,2v_{n-2}\}.$$

Then G' is an $(n-2)$-regular graph of order p. Hence, $ex(p;K_{1,n-1}) \ge e(G') = \frac{(n-2)p}{2} = [\frac{(n-2)p}{2}]$.

Case 4. $2 \nmid p$ and $2 \nmid n$. As $2 \mid n+1$, we can construct an $(n-1)$-regular graph G_1 of order p by using the argument in Case 3. Let

$$M_1=\begin{cases}\{23,45,\dots,(2k-2)(2k-1),k(2k)\} & \text{if } 2 \mid k-\frac{n+1}{2},\\ \{2(2k),3(k+3-\dfrac{n+1}{2}),45,67,\dots,(2k-2)(2k-1)\} & \text{if } 2\nmid k-\frac{n+1}{2}.\end{cases}$$

It is easily seen that $M_1 \subset G_1$. Set $G_2 = G_1 - M_1$. Then for $i = 2,3,\dots,2k$ we have

$$d_{G_2}(i) = \begin{cases} n-3 & \text{if } 2 \mid k-\frac{n+1}{2} \text{ and } i=k, \text{ or if } 2 \nmid k-\frac{n+1}{2} \text{ and } i=k+3-\frac{n+1}{2}, \\ n-2 & \text{otherwise.} \end{cases}$$

Thus G_2 does not contain $K_{1,n-1}$ and

$$2e(G_2) = \sum_{i=2}^{2k} d_{G_2}(i) = n-3+(2k-2)(n-2) = (n-2)p-1.$$

Hence $ex(p;K_{1,n-1}) \ge e(G_2) = \frac{(n-2)p-1}{2} = [\frac{(n-2)p}{2}]$.

Putting all the above together we prove the theorem. □

Corollary 2.1. *Let $k,p \in \mathbb{N}$ with $p \ge k+2$. Then there exists a $k-$regular graph of order p if and only if $2 \mid kp$.*

Proof. If G is a $k-$regular graph of order p, then $kp = 2e(G)$ and so $2 \mid kp$. If $2 \mid kp$, by the proof of Theorem 2.1 we know that there exists a $k-$regular graph of order p. □

Remark 2.1. In [4] Kirkman showed that K_{2n} is 1-factorable. In [5] Petersen proved that a graph G is 2-factorable if and only if G is $2p$-regular. Thus, Corollary 2.1 can be deduced from [4] and [5].

3. The Evaluation of $ex(p;T_n)$

Theorem 3.1. *Let $p,n \in \mathbb{N}$ with $p \ge n \ge 5$. Let $r \in \{0,1,\dots,n-2\}$ be given by $p \equiv r \pmod{n-1}$. Then*

$$ex(p;T_n) = \begin{cases} \Big[\dfrac{(n-2)(p-1)-r-1}{2}\Big] & \text{if } n \ge 7 \text{ and } 2 \le r \le n-4, \\ \dfrac{(n-2)p-r(n-1-r)}{2} & \text{otherwise.} \end{cases}$$

Proof. Let G be an extremal graph of order p not containing T_n. Suppose $v_0 \in V(G)$ and G_0 is the component of G such that $v_0 \in V(G_0)$. If $d(v_0) = m \ge n-1$, as G does not contain T_n we see that G_0 is a copy of $K_{1,m}$. Suppose $m+1 = k'(n-1)+r'$ with $k' \in \mathbb{N}$ and $r' \in \{0,1,\dots,n-2\}$. Then $k'K_{n-1} \cup K_{r'}$ does not contain T_n. As $\frac{n-2}{2} > 1$ and $\binom{r'}{2} - (r'-1) = \frac{(r'-1)(r'-2)}{2} \ge 0$, we find

$$e(k'K_{n-1} \cup K_{r'}) = k'\binom{n-1}{2} + \binom{r'}{2} > k'(n-1)+r'-1 = m = e(K_{1,m}) = e(G_0).$$

Hence $G_0 \notin Ex(m+1;T_n)$ and so $G \notin Ex(p;T_n)$. This contradicts the assumption. Therefore $d(v_0) \le n-2$ and so $\Delta(G) \le n-2$. If $d(v_0) = n-2$, as G_0 is an extremal graph not containing T_n we see that G_0 is a copy of K_{n-1}.

Suppose $p = k(n-1)+r$. Then $k \in \mathbb{N}$. From the above we may assume $G = sK_{n-1} \cup G_1$ with $s \in \{0,1,\dots,k\}$ and $\Delta(G_1) \le n-3$. If $s = k$, then clearly $G_1 = K_r$ and so $e(G) = k\binom{n-1}{2} + \binom{r}{2}$. If $s \le k-1$, as $\Delta(G_1) \le n-3$ implies G_1 does not contain any copies of T_n, we see that $G_1 \in Ex((k-s)(n-1)+r; K_{1,n-2})$. By Theorem 2.1 we have $e(G_1) = [\frac{(n-3)((k-s)(n-1)+r)}{2}]$. Hence

$$e(G) = e(sK_{n-1} \cup G_1) = s\binom{n-1}{2} + \Big[\frac{(n-3)((k-s)(n-1)+r)}{2}\Big].$$

Set $f(x) = x\binom{n-1}{2} + [\frac{(n-3)((k-x)(n-1)+r)}{2}]$. Then

$$\begin{aligned} f(x+1) &= (x+1)\binom{n-1}{2} + \Big[\frac{(n-3)((k-x)(n-1)+r)-(n-3)(n-1)}{2}\Big] \\ &= x\binom{n-1}{2} + \Big[\frac{(n-3)((k-x)(n-1)+r)}{2} + \frac{n-1}{2}\Big] > f(x). \end{aligned}$$

Thus, $f(k-1) > f(k-2) > \dots > f(0)$. Since G is an extremal graph, by the above we must have $s = k-1$ or k and so

$$\begin{aligned} &ex(p;T_n) = e(G) \\ &= \max\Big\{(k-1)\binom{n-1}{2} + \Big[\frac{(n-3)(n-1+r)}{2}\Big], k\binom{n-1}{2} + \binom{r}{2}\Big\}. \end{aligned}$$

Observe that

$$\frac{(n-3)(n-1+r)}{2} - \frac{r(r-1)}{2} - \frac{(n-1)(n-2)}{2} = \frac{r(n-2-r)-(n-1)}{2}.$$

We then have

$$ex(p;T_n) = k\binom{n-1}{2} + \binom{r}{2} + \max\Big\{0, \Big[\frac{r(n-2-r)-(n-1)}{2}\Big]\Big\}.$$

If $r \in \{1, n-3, n-2\}$, then clearly $[\frac{r(n-2-r)-(n-1)}{2}] < 0$. For $n = 6$ and $r = 2$ we also have $[\frac{r(n-2-r)-(n-1)}{2}] = -1 < 0$. Now assume $n \ge 7$ and $2 \le r \le n-4$. Then

$$\begin{aligned} r(n-2-r)-(n-1) &= \frac{n^2-8n+8}{4} - \Big(r - \frac{n-2}{2}\Big)^2 \\ &\ge \frac{n^2-8n+8}{4} - \Big(2 - \frac{n-2}{2}\Big)^2 = n-7 \ge 0 \end{aligned}$$

and so $[\frac{r(n-2-r)-(n-1)}{2}] \ge 0$. Hence

$$\begin{aligned} &ex(p;T_n) \\ &= \begin{cases} k\binom{n-1}{2} + \binom{r}{2} + [\frac{r(n-2-r)-(n-1)}{2}] & \text{if } n \ge 7 \text{ and } 2 \le r \le n-4, \\ k\binom{n-1}{2} + \binom{r}{2} & \text{otherwise.} \end{cases} \end{aligned}$$

To see the result, we note that $k\binom{n-1}{2}+\binom{r}{2}=\frac{(n-2)(p-r)+r^2-r}{2}=\frac{(n-2)p-r(n-1-r)}{2}$ and

$$k\binom{n-1}{2}+\binom{r}{2}+\Big[\frac{r(n-2-r)-(n-1)}{2}\Big]=\Big[\frac{(n-2)(p-1)-r-1}{2}\Big]. \qquad \square$$

4. The Evaluation of $ex(p;T_n^*)$

For $n\ge 4$ we recall that $T_n^*=(V,E)$ is the tree on n vertices with $V=\{v_0,v_1,\dots,v_{n-1}\}$ and $E=\{v_0v_1,\dots,v_0v_{n-3},v_{n-3}v_{n-2},v_{n-2}v_{n-1}\}$. Clearly $T_4^*=P_4$ and $T_5^*=P_5$.

Lemma 4.1. *Let $p,n\in\mathbb{N}$ with $p\ge n\ge 6$, and let $G\in Ex(p;T_n^*)$. Then $\Delta(G)\le n-2$.*

Proof. Suppose that $v_0\in V(G), d(v_0)=m\ge n-1$ and $\Gamma(v_0)=\{v_1,\dots,v_m\}$. Let G_0 be the component of G with $v_0\in V(G_0)$. If there are exactly t vertices $u_1,\dots,u_t\in V(G_0)$ such that $d(u_1,v_0)=\dots=d(u_t,v_0)=2$, then clearly $d(u_1)=\dots=d(u_t)=1$, $V(G_0)=\{v_0,v_1,\dots,v_m,u_1,\dots,u_t\}$ and $|V(G_0)|=1+m+t$. If $u_iv_j\notin E(G_0)$ for some $j\in\{1,2,\dots,m\}$ and every $i=1,2,\dots,t$, then clearly $d(v_j)\le 2$. Thus, $e(G_0)\le m+t+\frac{m}{2}$. Set $1+m+t=k(n-1)+r(0\le r<n-1)$. We see that

$$\begin{aligned}&k\binom{n-1}{2}+\binom{r}{2}-\frac{3m}{2}-t\\&=\frac{(n-2)(1+m+t-r)+r(r-1)-3m-2t}{2}\\&=\frac{(m+t)(n-5)-r(n-1-r)+(n-2)+t}{2}\\&\ge\frac{(n-1)(n-5)+(n-2)-r(n-1-r)}{2}\\&\ge\frac{(n-1)(n-5)+n-2-\frac{(n-1)^2}{4}}{2}=\frac{3(n-3)^2-16}{8}>0.\end{aligned}$$

Since $kK_{n-1}\cup K_r$ does not contain any copies of T_n^*, applying the above we deduce

$$e(G_0)\le\frac{3m+2t}{2}<k\binom{n-1}{2}+\binom{r}{2}=e(kK_{n-1}\cup K_r)\le ex(1+m+t;T_n^*).$$

As G is an extremal graph not containing T_n^*, we must have $e(G_0)=ex(1+m+t;T_n^*)$. This contradicts the above inequality $e(G_0)<ex(1+m+t;T_n^*)$. Hence the assumption $d(v_0)\ge n-1$ is not true. Thus $\Delta(G)\le n-2$. The proof is now complete. $\square$

Lemma 4.2. *Let $p,n\in\mathbb{N}$ with $p\ge n\ge 5$, and let $G\in Ex(p;T_n^*)$. Suppose that $v_0\in V(G), d(v_0)=n-2$ and G_0 is the component of G such that $v_0\in V(G_0)$. Then $G_0\cong K_{n-1}$.*

Proof. Suppose $\Gamma(v_0)=\{v_1,\dots,v_{n-2}\}$ and there are exactly t vertices $u_1,\dots,u_t\in V(G_0)$ such that $d(u_1,v_0)=\cdots=d(u_t,v_0)=2$. We first assume $t\ge 1$. Then clearly $d(u_1)=\cdots=d(u_t)=1$ and $V(G_0)=\{v_0,v_1,\dots,v_{n-2},u_1,\dots,u_t\}$. If $u_1v_i\in E(G)$ for some $i\in\{1,2,\dots,n-2\}$, then clearly $v_iv_j\notin E(G)$ for all $j\in\{1,2,\dots,n-2\}\setminus\{i\}$. Thus,

$$e(G_0)\le n-2+t+\binom{n-2-t}{2}\le\binom{n-2}{2}+t+1.$$

Assume $t=q(n-1)+t_0$ with $q\in\mathbb{Z}$ and $t_0\in\{0,1,\dots,n-2\}$. Then

$$\begin{aligned}&e((1+q)K_{n-1}\cup K_{t_0})-\binom{n-2}{2}-t-1\\&=(1+q)\binom{n-1}{2}+\binom{t_0}{2}-\binom{n-2}{2}-q(n-1)-t_0-1\\&=n-4+q\frac{(n-1)(n-4)}{2}+\frac{(t_0-1)(t_0-2)}{2}>0.\end{aligned}$$

As $(1+q)K_{n-1}\cup K_{t_0}$ does not contain T_n^*, applying the above we get

$$e(G_0)\le\binom{n-2}{2}+t+1<e((1+q)K_{n-1}\cup K_{t_0})\le ex(n-1+t;T_n^*).$$

Since G_0 is an extremal graph of order $n-1+t$ not containing T_n^*, we must have $e(G_0)=ex(n-1+t;T_n^*)$. This contradicts the above assertion. So $t\ge 1$ is not true and hence $V(G_0)=\{v_0,v_1,\dots,v_{n-2}\}$. As G_0 is an extremal graph not containing T_n^*, we see that $G_0\cong K_{n-1}$. This proves the lemma. □

Lemma 4.3. *Let $n,t\in\mathbb{N}$ with $n\ge 4$, and let $G\in Ex(n-2+t;T_n^*)$. Suppose that G is connected and $\Delta(G)=n-3$. Then $t\le n-4$ and $e(G)\le(n-3)^2$.*

Proof. Suppose $v_0\in V(G),d(v_0)=n-3,\Gamma(v_0)=\{v_1,\dots,v_{n-3}\}$ and $V(G)=\{v_0,v_1,\dots,v_{n-3},u_1,\dots,u_t\}$. Then $d(u_i,v_0)=2$ and $u_1,\dots,u_t$ must be independent. As G is connected and u_i is adjacent to some vertex in $\Gamma(v_0)$, we have

$$e(G)\le\sum_{i=1}^{n-3}d(v_i)\le\sum_{i=1}^{n-3}(n-3)=(n-3)^2.$$

On the other hand,

$$e(K_{n-1}\cup K_{n-4})=\frac{(n-1)(n-2)+(n-4)(n-5)}{2}=n^2-6n+11>(n-3)^2.$$

Thus, for $t\ge n-3$ we have

$$\begin{aligned}e(G)&=ex(n-2+t;T_n^*)\ge e\left(K_{n-1}\cup K_{n-4}\cup(t-(n-3))K_1\right)\\&=e(K_{n-1}\cup K_{n-4})>(n-3)^2.\end{aligned}$$

This contradicts the fact $e(G)\le(n-3)^2$. So $t\le n-4$. The proof is now complete. □

Lemma 4.4. *Let $p,n \in \mathbb{N}$ with $p \ge n \ge 4$, and $G \in Ex(p;T_n^*)$. Suppose $\Delta(G) \le n-3$. Then $p \le 2n-6$.*

Proof. Assume $p = 2n-4+t$. If $t \ge 2n$, we may write $t-2 = k(n-1)+r$, where $k \in \mathbb{N}$ and $r \in \{0,1,\dots,n-2\}$. Let $G_0 \in Ex(n-1+r;K_{1,n-3})$. From Theorem 2.1 we have $e(G_0) = [\frac{(n-1+r)(n-4)}{2}]$. Clearly $k(n-1) = t-2-r \ge 2n-2-r > r+1$. Thus,

$$\begin{aligned}
e((k+1)K_{n-1}\cup G_0) &= (k+1)\binom{n-1}{2} + \left[\frac{(n-1+r)(n-4)}{2}\right]\\
&\ge \frac{(k+1)(n-1)(n-2)+(n-1+r)(n-4)-1}{2}\\
&= \frac{((k+2)(n-1)+r)(n-3)}{2} + \frac{k(n-1)-r-1}{2}\\
&> \frac{((k+2)(n-1)+r)(n-3)}{2} = \frac{(n-3)p}{2}.
\end{aligned}$$

On the other hand, as $(k+1)K_{n-1}\cup G_0$ does not contain T_n^*, we have

$$e((k+1)K_{n-1}\cup G_0) \le ex(p;T_n^*) = e(G) \le \frac{(n-3)p}{2}.$$

This is a contradiction. Hence $t < 2n$.

If $t = 2n-1$, then $p = 2n-4+t = 3(n-1)+n-2$ and so

$$\frac{(n-3)p}{2} < e(3K_{n-1}\cup K_{n-2}) \le ex(p;T_n^*) = e(G) \le \frac{(n-3)p}{2}.$$

This is also a contradiction.

If $n-1 \le t < 2n-1$, setting $G_0 \in Ex(t-2;K_{1,n-3})$ and using Theorem 2.1 we see that

$$e(G_0) = ex(t-2;K_{1,n-3}) = \left[\frac{(n-4)(t-2)}{2}\right].$$

It is clear that $2K_{n-1}\cup G_0$ does not contain T_n^* as a subgraph and

$$\begin{aligned}
e(2K_{n-1}\cup G_0) &= 2\binom{n-1}{2} + \left[\frac{(n-4)(t-2)}{2}\right]\\
&\ge (n-1)(n-2) + \frac{(n-4)(t-2)-1}{2}\\
&= \frac{(2n-4+t)(n-3)}{2} + \frac{2n-1-t}{2} > \frac{(2n-4+t)(n-3)}{2}.
\end{aligned}$$

On the other hand,

$$e(2K_{n-1}\cup G_0) \le ex(2n-4+t;T_n^*) = e(G) \le \frac{(2n-4+t)(n-3)}{2}.$$

This is a contradiction.

By the above, we may assume $t \le n-2$. If $t = n-2$, then

$$\begin{aligned} ex(3n-6;T_n^*) &\ge e(2K_{n-1}\cup K_{n-4}) = \frac{2(n-1)(n-2)+(n-4)(n-5)}{2} \\ &> \frac{(3n-6)(n-3)}{2} \ge e(G) = ex(3n-6;T_n^*). \end{aligned}$$

This is a contradiction. If $t = n-3$, then

$$\begin{aligned} ex(3n-7;T_n^*) &\ge e(K_{n-1}\cup K_{n-3,n-3}) = \frac{(n-1)(n-2)}{2} + (n-3)^2 \\ &> \frac{(3n-7)(n-3)}{2} \ge e(G) = ex(3n-7;T_n^*). \end{aligned}$$

This is also a contradiction. Thus $t \ne n-2, n-3$.

Now we assume that $1 \le t \le n-4$. Suppose $H \in Ex(n-3;K_{1,n-3-t})$ and $V(H) = \{v_1,\dots,v_{n-3}\}$. We construct a graph $G_0 = (V(G_0),E(G_0))$ of order $n-3+t$ by defining $V(G_0) = \{u_1,\dots,u_t\}\cup V(H)$ and $E(G_0) = \{u_iv_j : 1 \le i \le t, 1 \le j \le n-3\} \cup E(H)$. It is easily seen that $d_{G_0}(v_i) \le n-4 (1 \le i \le n-3)$ and so G_0 does not contain any copies of T_n^*. Hence,

$$\begin{aligned} e(K_{n-1}\cup G_0) &= \binom{n-1}{2} + e(G_0) \\ &\le ex(2n-4+t;T_n^*) = e(G) \le \frac{(2n-4+t)(n-3)}{2}. \end{aligned}$$

Using Theorem 2.1 we see that

$$\begin{aligned} e(G_0) &= (n-3)t + \left[\frac{(n-3)(n-4-t)}{2}\right] \\ &\ge (n-3)t + \frac{(n-3)(n-4-t)-1}{2} \\ &= \frac{(2n-4+t)(n-3)}{2} - \binom{n-1}{2} + \frac{1}{2} \\ &> \frac{(2n-4+t)(n-3)}{2} - \binom{n-1}{2}, \end{aligned}$$

this contradicts the above assertion.

By the above we have $t \le 0$ and so $p \le 2n-4$. If $p = 2n-4$, since $K_{n-1}\cup K_{n-3}$ does not contain T_n^* we have

$$\begin{aligned} ex(2n-4;T_n^*) &\ge e(K_{n-1}\cup K_{n-3}) = \frac{(n-1)(n-2)+(n-3)(n-4)}{2} \\ &> \frac{(2n-4)(n-3)}{2} \ge e(G) = ex(2n-4;T_n^*). \end{aligned}$$

This is a contradiction.

Now we assume $p = 2n-5$. It is clear that

$$e(K_{n-1} \cup K_{n-4}) = \frac{(n-1)(n-2)+(n-4)(n-5)}{2} = n^2-6n+11.$$

As $K_{n-1} \cup K_{n-4}$ does not contain T_n^*, we see that $n^2-6n+11 \le ex(2n-5;T_n^*) = e(G)$. If $\Delta(G) \le n-4$, then clearly $e(G) \le \frac{(2n-5)(n-4)}{2} < n^2-6n+11$. This is a contradiction. Hence, $\Delta(G) = n-3$. Suppose that G_1 is the component of G such that $\Delta(G_1) = n-3$. If $|V(G_1)| = n-2+s$ for some $s \in \{0,1,\dots,n-3\}$, by Lemma 4.3 we have $s \le n-4$. As G is an extremal graph we have $G \backslash G_1 \cong K_{n-3-s}$ and so

$$\begin{aligned} e(G) = e(G_1)+e(G\backslash G_1) &\le \frac{(n-2+s)(n-3)}{2} + \binom{n-3-s}{2} \\ &= \frac{1}{2}\Big(s-\frac{n-4}{2}\Big)^2 + \frac{7n^2-40n+56}{8} \\ &\le \frac{1}{2}\Big(\frac{n-4}{2}\Big)^2 + \frac{7n^2-40n+56}{8} = n^2-6n+9 < n^2-6n+11, \end{aligned}$$

this contradicts the above assertion $e(G) \ge n^2-6n+11$. Therefore $p \ne 2n-5$ and so $p \le 2n-6$, which completes the proof. □

Theorem 4.1. *Let $p,n \in \mathbb{N}$ with $p \ge n-1 \ge 5$, and let $p = k(n-1)+r$ with $k \in \mathbb{N}$ and $r \in \{0,1,\dots,n-2\}$. Then*

$$ex(p;T_n^*) = \begin{cases} \dfrac{(k-1)(n-1)(n-2)}{2} + ex(n-1+r;T_n^*) & \text{if } 1 \le r \le n-5; \\ \dfrac{(n-2)p-r(n-1-r)}{2} & \text{if } r \in \{0,n-4,n-3,n-2\}. \end{cases}$$

Proof. Suppose $m \in \mathbb{N}$ and $m \ge 2n-5$. We assert that

$$ex(m;T_n^*) = \frac{(n-1)(n-2)}{2} + ex(m-(n-1);T_n^*). \tag{4.1}$$

Assume $G \in Ex(m;T_n^*)$. From Lemma 4.1 we know that $\Delta(G) \le n-2$. As $m \ge 2n-5$, by Lemma 4.4 we have $\Delta(G) = n-2$. Using Lemma 4.2 we see that G has a component isomorphic to K_{n-1} and so (4.1) is true. From (4.1) we deduce that for $k \ge 2$,

$$\begin{aligned} &ex(p;T_n^*) - ex(n-1+r;T_n^*) \\ &= \sum_{s=1}^{k-1} \big\{ex((s+1)(n-1)+r;T_n^*) - ex(s(n-1)+r;T_n^*)\big\} = (k-1)\binom{n-1}{2}. \end{aligned}$$

This is also true for $k = 1$.

For $r = 0$, we have $ex(n-1+r; T_n^*) = e(K_{n-1}) = \binom{n-1}{2}$ and so

$$ex(p; T_n^*) = (k-1)\binom{n-1}{2} + \binom{n-1}{2} = k\binom{n-1}{2} = \frac{(n-2)p}{2}.$$

For $r \in \{n-4, n-3, n-2\}$ we have $n-1+r \geq 2n-5$ and so by (4.1)

$$\begin{aligned} ex(p; T_n^*) &= (k-1)\binom{n-1}{2} + ex(n-1+r; T_n^*) \\ &= (k-1)\binom{n-1}{2} + \binom{n-1}{2} + ex(r; T_n^*) = k\binom{n-1}{2} + e(K_r) \\ &= \frac{(n-2)(p-r)}{2} + \binom{r}{2} = \frac{(n-2)p - r(n-1-r)}{2} \end{aligned}$$

as asserted. The proof is now complete. □

Theorem 4.2. *Let $p, n \in \mathbb{N}$ with $p \geq n \geq 6$ and $p = k(n-1)+1$ with $k \in \mathbb{N}$. Then*

$$ex(p; T_n^*) = \frac{(n-2)(p-1)}{2}.$$

Proof. Let $G_0 \in Ex(n; T_n^*)$. If $\Delta(G_0) \leq n-3$, then $e(G_0) \leq \frac{(n-3)n}{2} < \frac{(n-1)(n-2)}{2}$. On the other hand, $e(G_0) = ex(n; T_n^*) \geq e(K_{n-1} \cup K_1) = \frac{(n-1)(n-2)}{2}$. This is a contradiction. Thus $\Delta(G_0) \geq n-2$. Applying Lemmas 4.1 and 4.2 we see that $G_0 \cong K_{n-1} \cup K_1$ and so $ex(n; T_n^*) = e(G_0) = \frac{(n-1)(n-2)}{2}$. Now applying Theorem 4.1 we obtain

$$ex(p; T_n^*) = \frac{(k-1)(n-1)(n-2)}{2} + ex(n; T_n^*) = k\binom{n-1}{2} = \frac{(n-2)(p-1)}{2}.$$

This is the result. □

Theorem 4.3. *Let $p, n \in \mathbb{N}$, $p \geq n \geq 7$ and $p = k(n-1)+n-5$ with $k \in \mathbb{N}$. Then*

$$ex(p; T_n^*) = \frac{(n-2)(p-2)}{2} + 1.$$

Proof. Let $G_0 \in Ex(2n-6; T_n^*)$. If $\Delta(G_0) \leq n-3$, then $e(G_0) \leq \frac{(n-3)(2n-6)}{2} = (n-3)^2$. As $K_{n-3,n-3}$ does not contain any copies of T_n^*, we see that $e(G_0) \geq e(K_{n-3,n-3}) = (n-3)^2$. Hence $e(G_0) = (n-3)^2$. If $\Delta(G_0) \geq n-2$, by Lemmas 4.1 and 4.2 we have $G_0 \cong K_{n-1} \cup K_{n-5}$ Thus, $e(G_0) = e(K_{n-1} \cup K_{n-5}) = \binom{n-1}{2} + \binom{n-5}{2} = n^2 - 7n + 16$. Since $(n-3)^2 = n^2 - 6n + 9 \geq n^2 - 7n + 16$, we see that $ex(2n-6;T_n^*)=(n-3)^2$. Now applying the above and Theorem 4.1 we deduce

$$\begin{aligned}ex(p;T_n^*)&=(k-1)\binom{n-1}{2}+ex(2n-6;T_n^*)=(k-1)\binom{n-1}{2}+(n-3)^2\\&=k\frac{(n-1)(n-2)}{2}+\frac{n^2-9n+16}{2}=\frac{(n-2)(p-2)}{2}+1.\end{aligned}$$

This is the result. □

Lemma 4.5. *Let $n,r\in\mathbb{N}$ with $n\ge 7$ and $r\le n-5$. Then there is an extremal graph $G\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$ such that $\Delta(G)=n-3$ and G is connected.*

Proof. Let $G\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$. Then $\Delta(G)\le n-3$. For $r=n-5$ we see that $K_{n-3,n-3}\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$. So the result is true.

Now we assume $r\le n-6$. Suppose $H\in Ex(n-3;K_{1,n-5-r})$ and $V(H)=\{v_1,\dots,v_{n-3}\}$. From Theorem 2.1 we know that $e(H)=ex(n-3;K_{1,n-5-r})=[\frac{(n-3)(n-6-r)}{2}]$. Now we construct a graph $G_0=(V(G_0),E(G_0))$ of order $n-1+r$ by defining $V(G_0)=\{u_0,\dots,u_{r+1}\}\cup V(H)$ and $E(G_0)=\{u_iv_j:\ 0\le i\le r+1,1\le j\le n-3\}\cup E(H)$. It is easily seen that $d_{G_0}(v_i)\le n-4(1\le i\le n-3)$, $\Delta(G_0)=n-3$ and so G_0 does not contain any copies of T_n^* and $K_{1,n-2}$. Thus, for any $G\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$,

$$e(G)\ge e(G_0)=(n-3)(r+2)+\left[\frac{(n-3)(n-6-r)}{2}\right]=\left[\frac{(n-3)(n-2+r)}{2}\right].$$

If $\Delta(G)\le n-4$, we must have $G\in Ex(n-1+r;K_{1,n-3})$ and so $e(G)=[\frac{(n-4)(n-1+r)}{2}]$ by Theorem 2.1. As G is an extremal graph and

$$\begin{aligned}\left[\frac{(n-3)(n-2+r)}{2}\right]&\ge\frac{(n-3)(n-2+r)-1}{2}=\frac{(n-4)(n-1+r)+r+1}{2}\\&>\frac{(n-4)(n-1+r)}{2}\ge\left[\frac{(n-4)(n-1+r)}{2}\right],\end{aligned}$$

by the above we must have $\Delta(G)=n-3$.

Now assume $\Delta(G)=n-3$. If G is connected, the result is true. Suppose that G is not connected. Let G_1 be a component of G with $\Delta(G_1)=n-3$ and $|V(G_1)|=n-1+r-s$. Then $1\le s\le r+1\le n-5$. As G is an extremal graph, we must have $G=G_1\cup K_s$. Thus,

$$e(G)=e(G_1)+\binom{s}{2}\le\left[\frac{(n-3)(n-1+r-s)}{2}\right]+\frac{s(s-1)}{2}.$$

On the other hand, $e(G)\ge e(G_0)=[\frac{(n-3)(n-2+r)}{2}]$. Therefore,

$$\left[\frac{(n-3)(n-2+r)}{2}\right]-\left[\frac{(n-3)(n-1+r-s)}{2}\right]-\frac{s(s-1)}{2}\le 0.$$

For $s \geq 2$ we have $(s-1)(n-3-s) = (s-2)(n-4-s)+n-5 \geq n-5$ and so

$$\Big[\frac{(n-3)(n-2+r)}{2}\Big] - \Big[\frac{(n-3)(n-1+r-s)}{2}\Big] - \frac{s(s-1)}{2}$$
$$\geq \Big[-\frac{s^2-(n-2)s+n-3}{2}\Big] = \Big[\frac{(s-1)(n-3-s)}{2}\Big] \geq \Big[\frac{n-5}{2}\Big] > 0.$$

This contradicts the previous inequality. Thus $s = 1$ and hence $e(G) = e(G_1) \leq [\frac{(n-3)(n-2+r)}{2}] = e(G_0)$. By the previous argument, $e(G) \geq e(G_0)$. Therefore $e(G) = e(G_0)$. As G_0 is connected and $\Delta(G_0) = n-3$, we see that the result is true. □

Lemma 4.6. *Let $n, r \in \mathbb{N}$ with $n \geq 11$ and $3 \leq r \leq n-5$. Then there is an extremal graph $G \in Ex(n-1+r; T_n^*)$ such that $\Delta(G) = n-3$ and G is connected. Moreover, $ex(n-1+r; T_n^*) = ex(n-1+r; \{K_{1,n-2}, T_n^*\})$.*

Proof. Let $G \in Ex(n-1+r; T_n^*)$. For $r = n-5$ let $G_0 = K_{n-3,n-3}$. For $r \leq n-6$ let G_0 be the graph constructed in the proof of Lemma 4.5. Then $\Delta(G_0) = n-3$ and G_0 does not contain any copies of T_n^*. Thus, $e(G) \geq e(G_0)$. For $r = n-5$ we have $e(G_0) = (n-3)^2$. For $r \leq n-6$ we have $e(G_0) = [\frac{(n-3)(n-2+r)}{2}]$. Since $(n-3)^2 \geq \frac{(n-3)(n-2+n-5)}{2}$, we always have $e(G) \geq [\frac{(n-3)(n-2+r)}{2}]$ for $r \leq n-5$.

If $\Delta(G) \geq n-2$, by Lemmas 4.1 and 4.2 we have $G \cong K_{n-1} \cup K_r$. Thus, $e(G) = \binom{n-1}{2} + \binom{r}{2}$. Since $3 \leq r \leq n-5$ and $n \geq 11$ we see that $(r-2)(n-4-r) \geq 4$ and so

$$\Big[\frac{(n-3)(n-2+r)}{2}\Big] - \binom{n-1}{2} - \binom{r}{2} = \Big[\frac{(r-2)(n-4-r)-4}{2}\Big] \geq 0.$$

Therefore $e(G) \leq e(G_0)$ and so $e(G) = e(G_0)$. Since $\Delta(G_0) = n-3$ and G_0 is connected, the result holds in this case.

Now we assume $\Delta(G) \leq n-3$. Then $G \in Ex(n-1+r; \{K_{1,n-2}, T_n^*\})$. Applying Lemma 4.5 we see that the result is true. Thus the lemma is proved. □

Lemma 4.7. *Let $n, r \in \mathbb{N}$ with $n \geq 7$ and $r \leq n-5$. Then*

$$ex(n-1+r; \{K_{1,n-2}, T_n^*\}) = (n-3)(r+2) + ex(n-3; \{K_{1,n-4-r}, T_{n-2-r}^*\}).$$

Moreover, for $r \geq \frac{n-7}{2}$ we have

$$ex(n-1+r; \{K_{1,n-2}, T_n^*\})$$
$$= (n-3)(r+2) + \max\Big\{(n-5-r)^2, \Big[\frac{(n-6-r)(n-3)}{2}\Big]\Big\}.$$

Proof. It is clear that $ex(2n-6; \{K_{1,n-2}, T_n^*\}) = e(K_{n-3,n-3}) = (n-3)^2$. So the result is true for $r = n-5$.

Now assume $r \le n-6$. By Lemma 4.5, we can choose a graph $G \in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$ so that $\Delta(G)=n-3$ and G is connected. Suppose $u_0 \in V(G), d(u_0)=n-3, \Gamma(u_0)=\{v_1,\ldots,v_{n-3}\}$ and $V(G)=\{v_1,\ldots,v_{n-3},u_0,u_1,\ldots,u_{r+1}\}$. Then $d(u_i,u_0)=2$ for $i=1,2,\ldots,r+1$ and $\{u_0,u_1,\ldots,u_{r+1}\}$ is an independent set. If $u_iv_j \notin E(G)$ for some $i\in\{1,2,\ldots,r+1\}$ and $j\in\{1,2,\ldots,n-3\}$, as G is an extremal graph we see that $v_jv_k \in E(G)$ for some $k\in\{1,2,\ldots,n-3\}-\{j\}$. Set $G_1=G-v_jv_k+u_iv_j$. Then clearly G_1 does not contain T_n^*, $e(G)=e(G_1)$, $\Delta(G_1)=n-3$ and G_1 is connected. Repeating the above step we see that there is an extremal graph $G'\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$ such that $V(G')=\{v_1,\ldots,v_{n-3},u_0,u_1,\ldots,u_{r+1}\}$, $\Gamma(u_i)=\{v_1,\ldots,v_{n-3}\}$ for $i=0,1,\ldots,r+1$, $\Delta(G')=n-3$ and G' is connected. It is easily seen that

$$e(G')=(n-3)(r+2)+e(G'[v_1,\ldots,v_{n-3}]).$$

Set $H=G'[v_1,\ldots,v_{n-3}]$. Since $\Delta(G')=n-3$ and $G'\in Ex(n-1+r;\{K_{1,n-2},T_n^*\})$, we see that $\Delta(H)\le n-5-r$ and $H\in Ex(n-3;\{K_{1,n-4-r},T_{n-2-r}^*\})$.

Now we assume $r\ge \frac{n-7}{2}$. If $\Delta(H)=n-5-r$, we may assume $d(v_1)=n-5-r$ and $\Gamma_H(v_1)=\{v_2,\ldots,v_{n-4-r}\}$. Since G' does not contain T_n^* and $d_{G'}(v_1)=n-3$, we see that $\{v_{n-3-r},\ldots,v_{n-3}\}$ is an independent set. As $r\le n-6$, by the above we have $e(H)\le \sum_{i=2}^{n-4-r} d_H(v_i)\le (n-5-r)^2$. Since $r\ge \frac{n-7}{2}$ we have $n-3\ge 2(n-5-r)$. Set $H'=K_{n-5-r,n-5-r}\cup(3r+9-n)K_1$. Then $|V(H')|=n-1+r$ and $e(H')=(n-5-r)^2$, $\Delta(H')=n-5-r$ and H' does not contain T_{n-2-r}^*. As G' is an extremal graph, by the above we must have $e(H)=e(H')=(n-5-r)^2$. If $\Delta(H)<n-5-r$, then clearly $H\in Ex(n-3;K_{1,n-5-r})$. Using Theorem 2.1 we see that $e(H)=ex(n-3;K_{1,n-5-r})=[\frac{(n-3)(n-6-r)}{2}]$. Therefore, $e(H)=\max\{(n-5-r)^2,[\frac{(n-3)(n-6-r)}{2}]\}$ and so

$$\begin{aligned}&ex(n-1+r;\{K_{1,n-2},T_n^*\})\\&=e(G)=e(G')=(n-3)(r+2)+\max\Big\{(n-5-r)^2,\Big[\frac{(n-3)(n-6-r)}{2}\Big]\Big\}.\end{aligned}$$

This completes the proof. □

Theorem 4.4. *Let $p,n\in\mathbb{N}$, $p\ge n\ge 11$, $r\in\{2,3,\ldots,n-6\}$ and $p\equiv r \pmod{n-1}$. Let $m\in\{0,1,\ldots,r+1\}$ be given by $n-3\equiv m \pmod{r+2}$. Then*

$$ex(p;T_n^*) =\begin{cases}[\frac{(n-2)(p-1)-2r-m-3}{2}] & \text{if } r\ge 4 \text{ and } 2\le m\le r-1,\\ \frac{(n-2)(p-1)-m(r+2-m)-r-1}{2} & \text{otherwise.}\end{cases}$$

Proof. Suppose $s=[\frac{n-3}{r+2}]$. Then $n-3=s(r+2)+m$. As $r+2<n-3$ we see that

$s \in \mathbb{N}$. We claim that

$$ex(n-1+r;\{K_{1,n-2},T_n^*\}) \\ = \frac{(n-3-m)(n-1+r+m)}{2} + \max\Big\{m^2, \Big[\frac{(r+2+m)(m-1)}{2}\Big]\Big\}. \tag{4.2}$$

When $s=1$ we have $n-5-r=m<r+2$ and so $\frac{n-7}{2}<r<n-5$. Thus applying Lemma 4.7 we have

$$\begin{aligned} &ex(n-1+r;\{K_{1,n-2},T_n^*\}) \\ &= (n-3)(r+2)+\max\Big\{(n-5-r)^2, \Big[\frac{(n-6-r)(n-3)}{2}\Big]\Big\} \\ &= \frac{(n-3-m)(n-1+r+m)}{2} + \max\Big\{m^2, \Big[\frac{(r+2+m)(m-1)}{2}\Big]\Big\}. \end{aligned}$$

So (4.2) holds.

From now on we assume $s \geq 2$. For $i=0,1,\dots,s-2$ we have $n-i(r+2)-5 \geq n-3-(s-2)(r+2)-2 \geq 2(r+2)-2 > r \geq 2$. Thus, by Lemma 4.7 we have

$$\begin{aligned} &ex(n-3+r+2-i(r+2);\{K_{1,n-i(r+2)-2},T^*_{n-i(r+2)}\}) \\ &= (r+2)(n-3-i(r+2)) \\ &\qquad + ex(n-3-i(r+2);\{K_{1,n-(i+1)(r+2)-2},T^*_{n-(i+1)(r+2)}\}). \end{aligned}$$

Hence

$$\begin{aligned} &ex(n-1+r;\{K_{1,n-2},T_n^*\}) - ex(2(r+2)+m;\{K_{1,m+r+3},T^*_{m+r+5}\}) \\ &= ex(n-3+r+2;\{K_{1,n-2},T_n^*\}) \\ &\qquad - ex(n-3-(s-2)(r+2);\{K_{1,n-(s-1)(r+2)-2},T^*_{n-(s-1)(r+2)}\}) \\ &= \sum_{i=0}^{s-2}\Big(ex(n-3+r+2-i(r+2);\{K_{1,n-i(r+2)-2},T^*_{n-i(r+2)}\}) \\ &\qquad\qquad - ex(n-3-i(r+2);\{K_{1,n-(i+1)(r+2)-2},T^*_{n-(i+1)(r+2)}\})\Big) \\ &= \sum_{i=0}^{s-2}(r+2)(n-3-i(r+2)). \end{aligned}$$

Set $n'=m+r+5$. As $r>m-2$ and $r\geq 2$, we have $\frac{n'-7}{2}<r\leq n'-5$ and $n'\geq r+5\geq 7$. Thus, by Lemma 4.7 we have

$$\begin{aligned} &ex(2(r+2)+m;\{K_{1,m+r+3},T^*_{m+r+5}\}) \\ &= ex(n'-1+r;\{K_{1,n'-2},T^*_{n'}\}) \\ &= (n'-3)(r+2)+\max\Big\{(n'-5-r)^2, \Big[\frac{(n'-6-r)(n'-3)}{2}\Big]\Big\} \end{aligned}$$

$$= (r+2)(n-3-(s-1)(r+2)) + \max\Big\{m^2, \Big[\frac{(m-1)(m+r+2)}{2}\Big]\Big\}.$$

Therefore,

$$\begin{aligned}&ex(n-1+r;\{K_{1,n-2},T_n^*\})\\&=\sum_{i=0}^{s-1}(r+2)(n-3-i(r+2)) + \max\Big\{m^2, \Big[\frac{(m-1)(m+r+2)}{2}\Big]\Big\}.\end{aligned}$$

As

$$\begin{aligned}&\sum_{i=0}^{s-1}(r+2)(n-3-i(r+2))\\&=(r+2)\Big((n-3)s-(r+2)\frac{(s-1)s}{2}\Big)=\frac{s(r+2)}{2}\big(2(n-3)-(s-1)(r+2)\big)\\&=\frac{(n-3-m)(n-1+r+m)}{2},\end{aligned}$$

from the above we see that (4.2) is also true for $s\ge 2$.

Observe that $\frac{(m+r+2)(m-1)}{2} = m^2 + \frac{(r-m)(m-1)-2}{2}$. For $m = 0,1,r,r+1$, we have $(r-m)(m-1)-2\le 0$. Now assume $2\le m\le r-1$. If $r = 3$, then $m = 2$ and so $(r-m)(m-1)-2 = -1 < 0$. If $r\ge 4$, then clearly $(r-m)(m-1)-2\ge 0$. Thus, by (4.2) and the above we obtain

$$\begin{aligned}&ex(n-1+r;\{K_{1,n-2},T_n^*\})\\&=\begin{cases}\dfrac{(n-3-m)(n-1+r+m)}{2}+\Big[\dfrac{(r+2+m)(m-1)}{2}\Big]\\ \qquad\qquad\text{if } r\ge 4 \text{ and } 2\le m\le r-1,\\ \dfrac{(n-3-m)(n-1+r+m)}{2}+m^2\\ \qquad\qquad\text{otherwise.}\end{cases}\end{aligned}\tag{4.3}$$

For $r = 2$ we have $m\le r+1\le 3$. Let $G\in Ex(n+1;T_n^*)$. If $\Delta(G)\ge n-2$, by Lemmas 4.1 and 4.2 we have $G = K_{n-1}\cup K_2$. Thus, $e(G) = \binom{n-1}{2}+1$. If $\Delta(G)\le n-3$, then $G\in Ex(n+1;\{K_{1,n-2},T_n^*\})$. Thus, applying (4.3) we have

$$\begin{aligned}&ex(n+1;T_n^*)\\&=\max\Big\{\frac{(n-1)(n-2)}{2}+1, ex(n+1;\{K_{1,n-2},T_n^*\})\Big\}\\&=\max\Big\{\frac{(n-1)(n-2)}{2}+1, \frac{(n-3-m)(n+1+m)}{2}+m^2\Big\}\\&=\frac{(n-3-m)(n+1+m)}{2}+m^2+\max\Big\{0,-\frac{(m-2)^2+n-11}{2}\Big\}\\&=\frac{(n-3-m)(n+1+m)}{2}+m^2.\end{aligned}$$

For $r \geq 3$, by Lemma 4.6 we have $ex(n-1+r;T_n^*) = ex(n-1+r;\{K_{1,n-2},T_n^*\})$. Thus applying (4.3) we obtain

$$ex(n-1+r;T_n^*) = \begin{cases} \frac{(n-3-m)(n-1+r+m)}{2} + [\frac{(r+2+m)(m-1)}{2}] \\ \qquad \text{if } r \geq 4 \text{ and } 2 \leq m \leq r-1, \\ \frac{(n-3-m)(n-1+r+m)}{2} + m^2 \\ \qquad \text{otherwise.} \end{cases} \tag{4.4}$$

By the previous argument, (4.4) is also true for $r = 2$.

Now suppose $p = k(n-1)+r$. Then $k \in \mathbb{N}$. Combining (4.4) with Theorem 4.1 we deduce the following result:

$$ex(p;T_n^*) = \begin{cases} (k-1)\binom{n-1}{2} + \frac{(n-3-m)(n-1+r+m)}{2} + \left[\frac{(r+2+m)(m-1)}{2}\right] \\ \qquad \text{if } r \geq 4 \text{ and } 2 \leq m \leq r-1, \\ (k-1)\binom{n-1}{2} + \frac{(n-3-m)(n-1+r+m)}{2} + m^2 \quad \text{otherwise.} \end{cases}$$

To see the result, we note that

$$(k-1)\binom{n-1}{2} + \frac{(n-3-m)(n-1+r+m)}{2} + \left[\frac{(r+2+m)(m-1)}{2}\right] = \left[\frac{(n-2)(p-1)-2r-m-3}{2}\right]$$

and

$$(k-1)\binom{n-1}{2} + \frac{(n-3-m)(n-1+r+m)}{2} + m^2 = \frac{(n-2)(p-1)-m(r+2-m)-r-1}{2}. \qquad \square$$

Corollary 4.1. *Suppose $p,n,r \in \mathbb{N}$, $p \geq n \geq 11$, $\frac{n-7}{2} < r \leq n-6$ and $p \equiv r \pmod{n-1}$. Then*

$$ex(p;T_n^*) = \begin{cases} \left[\frac{(n-2)(p-2)-r}{2}\right] & \text{if } \frac{n-4}{2} \leq r \leq n-7, \\ \frac{(n-2)(p-3)}{2} + 3 & \text{if } r = n-6, \\ \frac{(n-2)(2p-5)+7}{4} & \text{if } r = \frac{n-5}{2}, \\ \frac{(n-2)(p-2)}{2} + 1 & \text{if } r = \frac{n-6}{2}. \end{cases}$$

Proof. Clearly $r > \frac{n-7}{2} \ge 2$. Set $m = n-5-r$. Then $1 \le m < r+2$ and $n-3 \equiv m \pmod{r+2}$. It is evident that

$$2 \le m \le r-1 \iff \frac{n-4}{2} \le r \le n-7.$$

As $n \ge 11$ we see that $r \ge \frac{n-4}{2}$ implies $r \ge 4$. Now applying Theorem 4.4 we deduce that

$$ex(p;T_n^*) = \begin{cases} \Big[\frac{(n-2)(p-1)-2r-(n-5-r)-3}{2}\Big] = \Big[\frac{(n-2)(p-2)-r}{2}\Big] \\ \qquad \text{if } \frac{n-4}{2} \le r \le n-7, \\ \frac{(n-2)(p-1)-(n-5-r)(r+2-(n-5-r))-r-1}{2} \\ \qquad \text{if } r = n-6 \text{ or } [\frac{n-5}{2}]. \end{cases}$$

This yields the result. □

Corollary 4.2. *Suppose $p,n \in \mathbb{N}$, $p \ge n \ge 11$, $2 \nmid n$ and $p \equiv \frac{n-7}{2} \pmod{n-1}$. Then*

$$ex(p;T_n^*) = \frac{(n-2)(2p-3)+3}{4}.$$

Proof. Taking $r = \frac{n-7}{2}$ and $m = 0$ in Theorem 4.4 we derive the result. □

Corollary 4.3. *Suppose $p,n \in \mathbb{N}$, $p \ge n \ge 11$ and $(n-1) \mid (p-2)$. Then*

$$ex(p;T_n^*) = \begin{cases} ((n-2)(p-1)-6)/2 & \text{if } n \equiv 0 \pmod 2, \\ ((n-2)(p-1)-7)/2 & \text{if } n \equiv 1 \pmod 4, \\ ((n-2)(p-1)-3)/2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $m \in \{0,1,2,3\}$ be given by $n-3 \equiv m \pmod 4$. Then clearly $m = 1,2,3$ or 0 according as $n \equiv 0,1,2$ or $3 \pmod 4$. Now putting $r = 2$ in Theorem 4.4 and applying the above we obtain the result. □

Corollary 4.4. *Suppose $p,n \in \mathbb{N}$, $p \ge n \ge 11$ and $(n-1) \mid (p-3)$. Then*

$$ex(p;T_n^*) = \begin{cases} (n-2)(p-1)/2-2 & \text{if } n \equiv 3 \pmod 5, \\ (n-2)(p-1)/2-4 & \text{if } n \equiv 2,4 \pmod 5, \\ (n-2)(p-1)/2-5 & \text{if } n \equiv 0,1 \pmod 5. \end{cases}$$

Proof. Let $m \in \{0,1,2,3,4\}$ be given by $n-3 \equiv m \pmod 5$. Then clearly $m = 2,3,4,0$ or 1 according as $n \equiv 0,1,2,3$ or $4 \pmod 5$. Now putting $r = 3$ in Theorem 4.4 and applying the above we obtain the result. □

In a similar way, putting $r = 4$ in Theorem 4.4 we deduce the following result.

Corollary 4.5. *Suppose $p, n \in \mathbb{N}$, $p \ge n \ge 11$ and $(n-1) \mid (p-4)$. Then*

$$ex(p; T_n^*) = \begin{cases} (n-2)(p-1)/2 - 7 & \text{if } n \equiv 0 \pmod 6, \\ (n-2)(p-1)/2 - 5 & \text{if } n \equiv \pm 2 \pmod 6, \\ ((n-2)(p-1) - 13)/2 & \text{if } n \equiv \pm 1 \pmod 6, \\ ((n-2)(p-1) - 5)/2 & \text{if } n \equiv 3 \pmod 6. \end{cases}$$

Corollary 4.6. *Suppose $p \in \mathbb{N}$, $p \ge 11$, $r \in \{0, 1, \dots, 9\}$ and $p \equiv r \pmod{10}$. Then*

$$ex(p; T_{11}^*) = \begin{cases} (9p - r(10-r))/2 & \text{if } r \in \{0, 1, 7, 8, 9\}, \\ (9p-12)/2 & \text{if } r = 2, \\ (9p-19)/2 & \text{if } r = 3, \\ (9p-22)/2 & \text{if } r = 4, \\ (9p-21)/2 & \text{if } r = 5, \\ (9p-16)/2 & \text{if } r = 6. \end{cases}$$

Proof. The result follows from Theorems 4.1-4.3 and Corollaries 4.1-4.2. □

Theorem 4.5. *Let $p, n \in \mathbb{N}$ with $6 \le n \le 10$ and $p \ge n$, and let $r \in \{0, 1, \dots, n-2\}$ be given by $p \equiv r \pmod{n-1}$.*

(i) *If $n = 6, 7$, then $ex(p; T_n^*) = \frac{(n-2)p - r(n-1-r)}{2}$.*

(ii) *If $n = 8, 9$, then*

$$ex(p; T_n^*) = \begin{cases} \dfrac{(n-2)p - r(n-1-r)}{2} & \text{if } r \ne n-5, \\ \dfrac{(n-2)(p-2)}{2} + 1 & \text{if } r = n-5. \end{cases}$$

(iii) *If $n = 10$, then*

$$ex(p; T_n^*) = \begin{cases} 4p - r(9-r)/2 & \text{if } r \ne 4, 5, \\ 4p - 7 & \text{if } r = 5, \\ 4p - 9 & \text{if } r = 4. \end{cases}$$

Proof. For $r \in \{0, 1, n-5, n-4, n-3, n-2\}$ the result follows from Theorems 4.1, 4.2 and 4.3. Now assume $2 \le r \le n-6$. Then $r \ge 2 > \frac{n-7}{2}$. By Lemma 4.7 we have

$$\begin{aligned} & ex(n-1+r; \{K_{1,n-2}, T_n^*\}) \\ & = (n-3)(r+2) + \max\Big\{(n-5-r)^2, \Big[\frac{(n-6-r)(n-3)}{2}\Big]\Big\}. \end{aligned}$$

If $G \in Ex(n-1+r; T_n^*)$ and $\Delta(G) \ge n-2$, using Lemmas 4.1 and 4.2 we see that $G \cong K_{n-1} \cup K_r$. Thus,

$$ex(n-1+r; T_n^*) = \max\Big\{\binom{n-1}{2} + \binom{r}{2}, ex(n-1+r; \{K_{1,n-2}, T_n^*\})\Big\}$$

$$= \max\Big\{\binom{n-1}{2}+\binom{r}{2},(n-3)(r+2)$$
$$+\max\Big\{(n-5-r)^2,\Big[\frac{(n-6-r)(n-3)}{2}\Big]\Big\}\Big\}.$$

From this we deduce that

$$ex(7+2;T_8^*)=\binom{7}{2}+\binom{2}{2},\quad ex(8+2;T_9^*)=\binom{8}{2}+\binom{2}{2},$$
$$ex(8+3;T_9^*)=\binom{8}{2}+\binom{3}{2},\quad ex(9+2;T_{10}^*)=\binom{9}{2}+\binom{2}{2},$$
$$ex(9+3;T_{10}^*)=\binom{9}{2}+\binom{3}{2},\quad ex(9+4;T_{10}^*)=43.$$

Suppose $p=k(n-1)+r$. Then $k\in\mathbb{N}$. By Theorem 4.1,

$$ex(p;T_n^*)=(k-1)\binom{n-1}{2}+ex(n-1+r;T_n^*)$$
$$=\frac{(n-2)(p-r)}{2}+ex(n-1+r;T_n^*)-\binom{n-1}{2}.$$

Now combining all the above we deduce the result. □

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