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On the properties of invariant functions

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Abstract

If $f(x,y)$ is a real function satisfying $y>0$ and $\sum_{r=0}^{n-1} f(x+ry,ny)=f(x,y)$ for $n=1,2,3,\ldots$, we say that $f(x,y)$ is an invariant function. Many special functions including Bernoulli polynomials, Gamma function and Hurwitz zeta function are related to invariant functions. In this paper we systematically investigate the properties of invariant functions.

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the set of integers and the set of positive integers, respectively. For $a\in\mathbb{Z}$ and $m\in\mathbb{Z}^+$ let $a(m)=a+m\mathbb{Z}=\{a+km \bigm| k\in\mathbb{Z}\}$. If

$$a_1(n_1)\cup a_2(n_2)\cup\cdots\cup a_k(n_k)=\mathbb{Z} \quad\text{and}\quad a_i(n_i)\cap a_j(n_j)=\emptyset \quad\text{for any}\quad i\ne j,$$

we say that $\{a_1(n_1),\ldots,a_k(n_k)\}$ is a disjoint covering system. In [9-11], Z.W. Sun showed that if $\{a_1(n_1),\ldots,a_k(n_k)\}$ is a disjoint covering system and $F(x,y)$ satisfies

$$\sum_{r=0}^{n-1} F\Big(\frac{x+r}{n},ny\Big)=F(x,y)\quad\text{for}\quad n=1,2,3,\ldots, \tag{1.1}$$

then $\sum_{s=1}^k F(\frac{x+a_s}{n_s},n_sy)=F(x,y)$. If $F(x,y)$ satisfies (1.1), $T\in\mathbb{Z}^+$ and $g(x+T)=g(x)$ for any $x\in\mathbb{Z}$, in [10] Z.W. Sun showed that

$$\sum_{r=0}^{nT-1} F\Big(\frac{r}{nT},nT\Big)g(x-r)=\sum_{r=0}^{T-1} F\Big(\frac{r}{T},T\Big)g(x-r)\quad\text{for any } n\in\mathbb{Z}^+.$$

In [11], Z.W. Sun also gave some examples of $F(x,y)$ satisfying (1.1). As we will see, many special functions can be used to construct functions satisfying (1.1).

If $f(x,y)$ is a real function with $y>0$ and

$$\sum_{r=0}^{n-1} f(x+ry,ny)=f(x,y)\quad\text{for}\quad n=1,2,3,\ldots, \tag{1.2}$$

we say that $f(x,y)$ is an invariant function. This is because

$$f(x+y,y)-f(x,y)=\sum_{r=0}^{n-1} f\big(x+\frac{y}{n}+\frac{r}{n}y,y\big)-\sum_{r=0}^{n-1} f\big(x+\frac{r}{n}y,y\big)=f\big(x+\frac{y}{n},\frac{y}{n}\big)-f\big(x,\frac{y}{n}\big)$$

for any $n\in\mathbb{Z}^+$ and so

$$(1.3)\quad f(x+y,y)-f(x,y)=\lim_{n\to+\infty}\Big(f\big(x+\frac{y}{n},\frac{y}{n}\big)-f\big(x,\frac{y}{n}\big)\Big)=\lim_{a\to 0^+}(f(x+a,a)-f(x,a)).$$

If the invariant function $f(x,y)$ is an integrable function of x at any closed intervals, we say that $f(x,y)$ is an integrable invariant function. The set of integrable invariant functions is denoted by I. If $F(x,y)$ is a real function satisfying (1.1) and $f(x,y)=F(x/y,y)$, then clearly f is an invariant function.

Let $[x]$ be the greatest integer not exceeding x. By the Hermite identity $\sum_{r=0}^{n-1}[x+\frac{r}{n}]=[nx]$ for $n\in\mathbb{Z}^+$, $[\frac{x}{y}]$ is an invariant function. The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are given by

$$B_0=1,\quad \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\quad(n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\ (n\ge 0).$$

Raabe's theorem states (see [7]) that $\sum_{r=0}^{n-1}B_m(x+\frac{r}{n})=n^{1-m}B_m(nx)\quad(m\ge 0,n\ge 1)$. Thus, $y^{m-1}B_m(\frac{x}{y})$ is an invariant function. More generally, if a is a real number and $F(x)$ is a real function satisfying

$$(1.4)\qquad \sum_{r=0}^{n-1}F\Big(x+\frac{r}{n}\Big)=n^{-a}F(nx)\quad\text{for any}\quad n\in\mathbb{Z}^+$$

and $f(x,y)=y^aF(\frac{x}{y})$, then for any $n\in\mathbb{Z}^+$,

$$\sum_{r=0}^{n-1}f(x+ry,ny)=\sum_{r=0}^{n-1}(ny)^aF\Big(\frac{x}{ny}+\frac{r}{n}\Big)=(ny)^a\cdot n^{-a}F\Big(\frac{x}{y}\Big)=f(x,y).$$

This shows that $f(x,y)$ is an invariant function. We note that the functional equation (1.4) has been investigated by several mathematicians including Bass, Kubert, Milnor and Walum. See [2],[5],[8] and [12]. It should also be mentioned that (1.4) can be viewed as a distribution relation. See [6, pp. 35-36,65-66].

In this paper, we systematically investigate the properties of invariant functions. In Section 2, we point out basic properties and more examples of invariant functions. In Section 3, we prove some interesting results for integrable invariant functions. In particular, if $f(x,y)$ is an integrable invariant function and $\frac{\partial f}{\partial y}$ exists, then

$$f(x,y)=\int_x^{x-y}\frac{\partial}{\partial y}f(t,y)\,dt;$$

if g and h are integrable invariant functions and

$$g*h(x,y)=\int_0^x g(t,y)h(x-t,y)dt+\int_x^y g(t,y)h(x+y-t,y)dt,$$

then $g*h$ is also an integrable invariant function and

$$\int_0^y g*h(x,y)dx = \int_0^y g(x,y)dx \cdot \int_0^y h(x,y)dx.$$

For $m,n\in\mathbb{Z}^+$ we have

$$B_{m+n}(x) = -\binom{m+n}{m}\Big(\int_0^1 B_m(x-t)B_n(t)dt + m\int_x^1 (x-t)^{m-1}B_n(t)dt\Big)$$

and so

$$\frac{-y^{m-1}B_m(\frac{x}{y})}{m!} * \frac{-y^{n-1}B_n(\frac{x}{y})}{n!} = \frac{-y^{m+n-1}B_{m+n}(\frac{x}{y})}{(m+n)!}.$$

2. Basic properties and examples of invariant functions

Let $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{C}$ be the set of real numbers, the set of positive real numbers and the set of complex numbers, respectively. From the definition of invariant function one can easily prove Propositions 2.1-2.4.

Proposition 2.1. *Suppose that $f(x,y)$ is an invariant function, $a,b,c\in\mathbb{R}$, $c>0$ and $F(x,y)=af(b+cx,cy)$, then $F(x,y)$ is also an invariant function.*

Proposition 2.2. *If f is an invariant function and $\frac{\partial f}{\partial x}$ exists, then $\frac{\partial f}{\partial x}$ is also an invariant function.*

Proposition 2.3. *Let $F(x,y)$ be the mapping from $\mathbb{R}\times\mathbb{R}^+$ to $\mathbb{C}$, and $F(x,y)=f(x,y)+ig(x,y)$, where $f(x,y)$ is the real part of $F(x,y)$. If $\sum_{r=0}^{n-1}F(x+ry,ny)=F(x,y)$ for any $n\in\mathbb{Z}^+$, then $f(x,y)$ and $g(x,y)$ are invariant functions.*

Proposition 2.4. *Suppose that $f(x,y)$ is an invariant function and $m,n\in\mathbb{Z}^+$. Then*

$$\sum_{r=0}^{n-1} f(x+rmy,ny) = \sum_{r=0}^{m-1} f(x+rny,my).$$

Proposition 2.5. *If f is an invariant function and $F(x,y)=f(y-x,y)$, then F is also an invariant function.*

Proof. For $n\in\mathbb{Z}^+$,

$$\sum_{r=0}^{n-1}F(x+ry,ny) = \sum_{r=0}^{n-1} f(ny-(x+ry),ny) = \sum_{r=0}^{n-1} f(y-x+(n-1-r)y,ny)$$
$$= \sum_{k=0}^{n-1} f(y-x+ky,ny) = f(y-x,y) = F(x,y).$$

This proves the proposition.

For $x\in\mathbb{R}$ let $\{x\}$ be the fractional part of x. That is, $\{x\}=x-[x]$.

Proposition 2.6. *Suppose that f is an invariant function and $t\in\mathbb{R}$. Let*

$$f_1(x,y) = f\Big(y\Big\{\frac{t+x}{y}\Big\},y\Big) \quad \text{and} \quad f_2(x,y) = f\Big(y\Big\{\frac{t-x}{y}\Big\},y\Big).$$

Then both f_1 and f_2 are invariant functions.

Proof. For $n \in \mathbb{Z}^+$ we have

$$\begin{aligned}
&\sum_{r=0}^{n-1} f_1(x+ry, ny) \\
&= \sum_{r=0}^{n-1} f\Big(ny\Big\{\frac{t+x+ry}{ny}\Big\}, ny\Big) = \sum_{r=0}^{n-1} f\Big(ny\Big\{\frac{r+[\frac{t+x}{y}]+\{\frac{t+x}{y}\}}{n}\Big\}, ny\Big) \\
&= \sum_{r=0}^{n-1} f\Big(ny\Big\{\frac{r+\{\frac{t+x}{y}\}}{n}\Big\}, ny\Big) = \sum_{r=0}^{n-1} f\Big(y\{\frac{t+x}{y}\}+ry, ny\Big) = f\Big(y\Big\{\frac{t+x}{y}\Big\}, y\Big).
\end{aligned}$$

This shows that $f_1(x,y)$ is an invariant function. Since $f_2(x,y) = f_1(y-x,y)$, $f_2(x,y)$ is also an invariant function by Proposition 2.5. This completes the proof.

Proposition 2.7. *Let $h(x)$ be a real function, and let*

$$f(x,y) = \frac{1}{y}\sum_{k=1}^{\infty} h\Big(\frac{k}{y}\Big)\mathrm{cos}2\pi\frac{kx}{y} \quad \textit{and} \quad g(x,y) = \frac{1}{y}\sum_{k=1}^{\infty} h\Big(\frac{k}{y}\Big)\mathrm{sin}2\pi\frac{kx}{y}.$$

Then f and g are invariant functions.

Proof. Suppose $F(x,y) = \frac{1}{y}\sum_{k=1}^{\infty} h(\frac{k}{y})\mathrm{e}^{2\pi i\frac{kx}{y}}$. For $n \in \mathbb{Z}^+$ we see that

$$\begin{aligned}
\sum_{r=0}^{n-1} F(x+ry, ny) &= \sum_{r=0}^{n-1} \frac{1}{ny}\sum_{k=1}^{\infty} h\Big(\frac{k}{ny}\Big)\mathrm{e}^{2\pi i\frac{k(x+ry)}{ny}} \\
&= \frac{1}{ny}\sum_{k=1}^{\infty} h\Big(\frac{k}{ny}\Big)\mathrm{e}^{2\pi i\frac{kx}{ny}}\sum_{r=0}^{n-1}\mathrm{e}^{2\pi i\frac{kr}{n}} \\
&= \frac{1}{ny}\sum_{m=1}^{\infty} h\Big(\frac{m}{y}\Big)\mathrm{e}^{2\pi i\frac{mx}{y}} \cdot n = F(x,y).
\end{aligned}$$

By Euler's formula $\mathrm{e}^{i\theta} = \mathrm{cos}\theta + i\,\mathrm{sin}\theta$, we have $F(x,y) = f(x,y) + ig(x,y)$. Thus the result follows from Proposition 2.3.

Proposition 2.8. *Let $f(x,y)$ be a real function with $y > 0$. Then $f(x,y)$ is an invariant function if and only if (1.2) holds for $0 < x \le y$ and (1.3) holds for all $x \in \mathbb{R}$.*

Proof. If $f(x,y)$ is an invariant function, then clearly (1.2) holds for $0 < x \le y$ and (1.3) holds for all $x \in \mathbb{R}$ by Section 1. Now suppose that (1.2) holds for $0 < x \le y$ and (1.3) holds for all $x \in \mathbb{R}$. Since $\sum_{r=0}^{n-1} f(x+ry,ny) = f(x,y)$ $(n = 1,2,3,\ldots)$ implies that

$$\begin{aligned}
\sum_{r=0}^{n-1} f(x+y+ry, ny) &= f(x+ny,ny) - f(x,ny) + \sum_{r=0}^{n-1} f(x+ry,ny) \\
&= f(x+y,y) - f(x,y) + f(x,y) = f(x+y,y)\ (n = 1,2,3,\ldots)
\end{aligned}$$

and

$$\sum_{r=0}^{n-1} f(x-y+ry, ny) = f(x-y,ny) - f(x-y+ny,ny) + \sum_{r=0}^{n-1} f(x+ry,ny)$$

$$= f(x-y,y) - f(x,y) + f(x,y) = f(x-y,y) \ (n=1,2,3,\ldots),$$

we see that $f(x,y)$ is an invariant function, which completes the proof.

Proposition 2.9. *Suppose that $g(x)$ and $h(x)$ are real functions with $g(1)=1$ and $h(0)(h(2)-h(1)) \neq 0$, and $f(x,y) = g(y)h(\frac{x}{y})$ for $x \in \mathbb{R}$ and $y > 0$. Then $f(x,y)$ is an invariant function if and only if the following conditions hold:*

(i) $$g(y) = \frac{h(y+1)-h(y)}{h(2)-h(1)},$$

(ii) $$\sum_{r=0}^{n-1} h\Big(\frac{x+r}{n}\Big) = \frac{h(2)-h(1)}{h(n+1)-h(n)} h(x) \quad \text{for } n \in \mathbb{Z}^+,$$

(iii) $$g(ny) = g(n)g(y) \quad \text{for } n \in \mathbb{Z}^+.$$

Proof. We first assume that $f(x,y)$ is an invariant function. Clearly, $f(x+y,y) - f(x,y) = (h(\frac{x}{y}+1) - h(\frac{x}{y}))g(y)$. By (1.3), the value of $f(x+y,y) - f(x,y)$ does not depend on the choice of y. Hence, $(h(\frac{x}{y}+1) - h(\frac{x}{y}))g(y) = (h(x+1)-h(x))g(1)$. Taking $x = y$ yields (i). For $n \in \mathbb{Z}^+$ we see that

$$\sum_{r=0}^{n-1} h\Big(\frac{x+ry}{ny}\Big) g(ny) = h\Big(\frac{x}{y}\Big) g(y) \quad \text{and so} \quad \sum_{r=0}^{n-1} h\Big(\frac{x/y+r}{n}\Big) = h\Big(\frac{x}{y}\Big) \frac{g(y)}{g(ny)}.$$

Taking $y = 1$ and then applying (i) gives

$$\sum_{r=0}^{n-1} h\Big(\frac{x+r}{n}\Big) = \frac{g(1)}{g(n)} h(x) = \frac{h(2)-h(1)}{h(n+1)-h(n)} h(x).$$

This proves (ii). By the above, for $n \in \mathbb{Z}^+$, $\sum_{r=0}^{n-1} h(\frac{r}{n}) = h(0)\frac{g(y)}{g(ny)}$. Thus,

$$\frac{g(y)}{g(ny)} = \frac{\sum_{r=0}^{n-1} h(\frac{r}{n})}{h(0)} = \frac{g(1)}{g(n)} \quad \text{and so} \quad g(ny) = g(n)g(y).$$

Conversely, if $f(x,y)$ satisfies (i),(ii) and (iii), for $n \in \mathbb{Z}^+$ we have

$$\sum_{r=0}^{n-1} f(x+ry,ny) = g(ny) \sum_{r=0}^{n-1} h\Big(\frac{x/y+r}{n}\Big) = g(n)g(y) \cdot \frac{h(2)-h(1)}{h(n+1)-h(n)} h\Big(\frac{x}{y}\Big)$$
$$= g(y)h\Big(\frac{x}{y}\Big) = f(x,y).$$

This shows that $f(x,y)$ is an invariant function, which completes the proof.

Remark 2.1 If $g(x)$ is a continuous function for $x > 0$ satisfying $g(1) = 1$ and $g(nx) = g(n)g(x)$ for $n \in \mathbb{Z}^+$, one may show that $g(x) = x^a$ for some $a \in \mathbb{R}$.

Now we list some typical examples of invariant functions.

Example 2.1. $\frac{1}{y} \in I$.

Example 2.2. $y^{m-1}B_m(\frac{x}{y}) \in I$ $(m \in \mathbb{Z}^+)$.

Let $\{E_n(x)\}$ be the Euler polynomials given by $E_n(x) = \frac{1}{2^n}\sum_{k=0}^{[n/2]} \binom{n}{2k}(2x-1)^{n-2k}E_{2k}$, where $\{E_{2k}\}$ are the Euler numbers defined by $E_0 = 1$ and $E_{2k} = -\sum_{r=1}^{k}\binom{2k}{2r}E_{2k-2r}$

$(k \geq 1)$. From [7, p.30], $E_m(x) = n^m \sum_{r=0}^{n-1}(-1)^r E_m(\frac{x+r}{n})$. Thus, for $x, y \in \mathbb{Z}$ and $m, n \in \mathbb{Z}^+$ with $2 \nmid ny$, setting $f(x,y) = (-1)^x y^m E_m(\frac{x}{y})$ we have

$$\sum_{r=0}^{n-1} f(x+ry, ny) = \sum_{r=0}^{n-1}(-1)^{x+ry}(ny)^m E_m\Big(\frac{x+ry}{ny}\Big) = (-1)^x y^m E_m\Big(\frac{x}{y}\Big) = f(x,y).$$

Example 2.3. $[\frac{x}{y}] \in I$ and $\{\frac{x}{y}\} - \frac{1}{2} \in I$.

Since $[\frac{x}{y}] \in I$ and $\frac{x}{y} - \frac{1}{2} = B_1(\frac{x}{y}) \in I$ we have $\{\frac{x}{y}\} - \frac{1}{2} \in I$.

Example 2.4. For $a \in \mathbb{R}$ let

$$f(x,y) = \begin{cases} 1 & \text{if } \dfrac{a-x}{y} \in \mathbb{Z}, \\ 0 & \text{if } \dfrac{a-x}{y} \notin \mathbb{Z}. \end{cases}$$

Then $f \in I$. This is because $(a-x)/y \in \mathbb{Z}$ implies that there is a unique $r \in \{0, 1, \dots, n-1\}$ such that $(a-x)/y \equiv r \pmod{n}$ for $n \in \mathbb{Z}^+$.

Example 2.5. For $a > 0$ and $a \neq 1$, we have $\frac{a^x}{a^y - 1} \in I$.

This is because

$$(2.1) \qquad \sum_{r=0}^{n-1} \frac{a^{x+ry}}{a^{ny}-1} = \frac{a^x}{a^{ny}-1} \sum_{r=0}^{n-1} a^{ry} = \frac{a^x}{a^y - 1} \quad \text{for} \quad n \in \mathbb{Z}^+.$$

Example 2.6. Let $r > 0$, $r \neq 1$ and $\theta \in \mathbb{R}$, and let

$$f(x,y) = \frac{r^{x+y}\text{cos}(x-y)\theta - r^x \text{cos}x\theta}{1 - 2r^y\text{cos}y\theta + r^{2y}} \quad \text{and} \quad g(x,y) = \frac{r^{x+y}\text{sin}(x-y)\theta - r^x\text{sin}x\theta}{1 - 2r^y\text{cos}y\theta + r^{2y}}.$$

Then $f(x,y), g(x,y) \in I$.

By Euler's formula,

$$\frac{(r\mathrm{e}^{i\theta})^x}{(r\mathrm{e}^{i\theta})^y - 1} = \frac{r^x(\text{cos}x\theta + i\ \text{sin}x\theta)}{r^y\text{cos}y\theta - 1 + i\ \text{sin}y\theta \cdot r^y} = \frac{r^x(\text{cos}x\theta + i\ \text{sin}x\theta)(r^y\text{cos}y\theta - 1 - i\ \text{sin}y\theta \cdot r^y)}{(r^y\text{cos}y\theta - 1)^2 + r^{2y}\text{sin}^2 y\theta}$$
$$= f(x,y) + ig(x,y).$$

Now, from Proposition 2.3 and (2.1) we deduce that $f(x,y), g(x,y) \in I$.

Example 2.7. Let $r > 0$, $r \neq 1$ and $f(x,y) = \log\,(1 - 2r^{\frac{1}{y}} \cos 2\pi\frac{x}{y} + r^{\frac{2}{y}})$. Then $f \in I$.

For $n \in \mathbb{Z}^+$ and $z \in \mathbb{C}$,

$$\prod_{j=0}^{n-1}(1 - z\mathrm{e}^{2\pi i\frac{j}{n}}) = \prod_{j=0}^{n-1}(-\mathrm{e}^{2\pi i\frac{j}{n}})(z - \mathrm{e}^{-2\pi i\frac{j}{n}}) = (-1)^n \mathrm{e}^{2\pi i\frac{1+2+\cdots+n-1}{n}}(z^n - 1) = 1 - z^n.$$

Set $z = r^{\frac{1}{ny}}\mathrm{e}^{2\pi i\frac{x}{ny}}$. We get $\prod_{j=0}^{n-1}\big(1 - r^{\frac{1}{ny}}\mathrm{e}^{2\pi i\frac{x+jy}{ny}}\big) = 1 - r^{\frac{1}{y}}\mathrm{e}^{2\pi i\frac{x}{y}}$. Thus,

$$\sum_{j=0}^{n-1} \log\ |1 - r^{\frac{1}{ny}}\mathrm{e}^{2\pi i\frac{x+jy}{ny}}| = \log|1 - r^{\frac{1}{y}}\mathrm{e}^{2\pi i\frac{x}{y}}|.$$

That is,

$$\sum_{j=0}^{n-1} \log\ \Big(1 - 2r^{\frac{1}{ny}} \cos 2\pi \frac{x+jy}{ny} + r^{\frac{2}{ny}}\Big) = \log\Big(1 - 2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}}\Big).$$

Example 2.8. Let $r > 0$, $r \neq 1$ and $f(x,y) = \frac{r^{\frac{1}{y}} \sin 2\pi \frac{x}{y}}{y(1-2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}})}$. Then $f \in I$.

This is immediate from Example 2.7 and Proposition 2.2.

Example 2.9. Let $0 < r < 1$ and $f(x,y) = \frac{1-r^{\frac{2}{y}}}{y(1-2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}})}$. Then $f \in I$.

Set

$$F(x,y) = \frac{1}{y(1 - r^{\frac{1}{y}} e^{2\pi i \frac{x}{y}})} - \frac{1}{2y} = \frac{1}{2y} + \frac{1}{y} \sum_{k=1}^{\infty} r^{\frac{k}{y}} e^{2\pi i \frac{kx}{y}}.$$

By Example 2.1 and the proof of Proposition 2.7, $\sum_{m=0}^{n-1} F(x+my, ny) = F(x,y)$ for $n \in \mathbb{Z}^+$. Note that

$$F(x,y) = \frac{1}{y(1 - r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} - i \sin 2\pi \frac{x}{y} \cdot r^{\frac{1}{y}})} - \frac{1}{2y} = \frac{1 - r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + i \sin 2\pi \frac{x}{y} \cdot r^{\frac{1}{y}}}{y(1 - 2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}})} - \frac{1}{2y}$$

$$= \frac{1 - r^{\frac{2}{y}}}{2y(1 - 2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}})} + i \frac{r^{\frac{1}{y}} \sin 2\pi \frac{x}{y}}{y(1 - 2r^{\frac{1}{y}} \cos 2\pi \frac{x}{y} + r^{\frac{2}{y}})}.$$

We see that $f(x,y) \in I$ by the above and Proposition 2.3.

Example 2.10 ([11]). For $y > 0$ let

$$f(x,y) = \begin{cases} \log \big|2 \sin \pi \frac{x}{y}\big| & \text{if } \frac{x}{y} \notin \mathbb{Z}, \\ -\log y & \text{if } \frac{x}{y} \in \mathbb{Z}. \end{cases}$$

Then $f \in I$.

Now we give a straightforward proof of Example 2.10. For $x \neq 2k\pi$ $(k \in \mathbb{Z})$ we know that $\log \big|2\sin \frac{x}{2}\big| = -\sum_{k=1}^{\infty} \frac{\cos kx}{k}$. Thus, for $\frac{x}{y} \notin \mathbb{Z}$ we have $\log \big|2 \sin \pi \frac{x}{y}\big| = -\frac{1}{y} \sum_{k=1}^{\infty} \frac{\cos 2k\pi \frac{x}{y}}{k/y}$. Since $\frac{x}{y} \notin \mathbb{Z}$ implies $\frac{x+ry}{ny} \notin \mathbb{Z}$ for $n \in \mathbb{Z}^+$ and $r \in \{0, 1, \ldots, n-1\}$, from the above and Proposition 2.7 we deduce that $\sum_{r=0}^{n-1} f(x+ry, ny) = f(x,y)$ for any $n \in \mathbb{Z}^+$. Now assume that $\frac{x}{y} \in \mathbb{Z}$. For $n \in \mathbb{Z}^+$ there is a unique $r \in \{0, 1, \ldots, n-1\}$ such that $\frac{x+ry}{ny} = \frac{x/y+r}{n} \in \mathbb{Z}$. Hence,

$$\sum_{r=0}^{n-1} f(x+ry, ny) = -\log ny + \sum_{\substack{r=0 \\ n \nmid (x/y+r)}}^{n-1} \log \big|2 \sin \pi \frac{x/y+r}{n}\big| = -\log ny + \sum_{k=1}^{n-1} \log 2 \sin \frac{k\pi}{n}.$$

Since

$$\prod_{k=1}^{n-1} 2 \sin \frac{k\pi}{n} = \prod_{k=1}^{n-1} \frac{e^{ik\pi/n} - e^{-ik\pi/n}}{i} = \prod_{k=1}^{n-1} \big(1 - e^{-\frac{2k\pi i}{n}}\big) = \lim_{x \to 1} \frac{x^n - 1}{x - 1} = n,$$

we get

$$\sum_{r=0}^{n-1} f(x+ry,ny) = -\log ny + \log \prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n} = -\log ny + \log n = -\log y = f(x,y).$$

Example 2.11 ([11]). For $y>0$ let

$$f(x,y) = \begin{cases} \frac{1}{y}\cot\pi\frac{x}{y} & \text{if } \frac{x}{y}\notin\mathbb{Z}, \\ 0 & \text{if } \frac{x}{y}\in\mathbb{Z}. \end{cases}$$

Then $f\in I$.

The Gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,dt \quad (x>0) \quad \text{and} \quad \Gamma(x+1)=x\Gamma(x) \quad (x\neq 0,-1,-2,\ldots).$$

For the properties of $\Gamma(x)$ see [1], [3] and [7].

Example 2.12 ([11]). For $x\in\mathbb{R}$ and $y>0$ let

$$f(x,y) = \begin{cases} \log\left|\frac{y^{\frac{x}{y}}\Gamma(\frac{x}{y})}{\sqrt{2\pi y}}\right| & \text{if } \frac{x}{y}\notin\{0,-1,-2,\ldots\}, \\ \log\frac{y^{\frac{x}{y}}}{(-\frac{x}{y})!}\sqrt{2\pi y} & \text{if } \frac{x}{y}\in\{0,-1,-2,\ldots\}. \end{cases}$$

Then $f\in I$.

For $s>1$ and $x\notin\{0,-1,-2,\ldots\}$ the Hurwitz zeta function $\zeta(s,x)$ is defined by

$$\zeta(s,x) = \sum_{n=0}^\infty \frac{1}{(n+x)^s}. \tag{2.2}$$

From [1, pp.51-52] we know that $\zeta(s,x)$ has a continuation to the whole complex plane with a simple pole at $s=1$. That is,

$$\zeta(s,x) = \frac{\Gamma(1-s)\mathrm{e}^{-i\pi s}}{2\pi i}\int_C \frac{z^{s-1}\mathrm{e}^{-xz}}{1-\mathrm{e}^{-z}}dz \quad \text{for} \quad s\neq 1, \tag{2.3}$$

where C starts at infinity on the positive real axis, encircles the origin once in the positive direction, excluding the points $2k\pi i$ $(k\in\mathbb{Z})$, and returns to positive infinity. In particular, for $Re\ s<0$ and $x\in(0,1]$, Hurwitz showed (see [3, p.26]) that

$$\zeta(s,x) = \frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\Big(\sin\frac{s\pi}{2}\sum_{k=1}^\infty k^{s-1}\cos 2\pi kx + \cos\frac{s\pi}{2}\sum_{k=1}^\infty k^{s-1}\sin 2\pi kx\Big). \tag{2.4}$$

From (2.2) and (2.3) one may easily deduce that

$$\zeta(s,x+1)-\zeta(s,x) = -x^{-s} \quad \text{for } s\neq 1. \tag{2.5}$$

It is well known (see [3, p.27]) that

$$\zeta(1-m,x) = -\frac{B_m(x)}{m} \quad \text{for} \quad m\in\mathbb{Z}^+. \tag{2.6}$$

Example 2.13. For $s < 0$, we have $y^{-s}\zeta(s, \frac{x}{y}) \in I$ by (2.4), (2.5) and Propositions 2.7-2.8. For $s > 1$ define $f(x, y) = y^{-s}\zeta(s, \frac{x}{y})$ for $y > 0$ and $\frac{x}{y} \notin \{0, -1, -2, \ldots\}$. Then $\sum_{r=0}^{n-1} f(x + ry, ny) = f(x, y)$ for any $n \in \mathbb{Z}^+$.

Example 2.14 Let $h(x)$ be a real function and $\sum_{k=0}^{\infty} h(x + ky) = f(x, y)$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$. Then f is an invariant function.

For $n \in \mathbb{Z}^+$ we see that

$$\sum_{r=0}^{n-1} f(x + ry, ny) = \sum_{r=0}^{n-1}\sum_{k=0}^{\infty} h(x + ry + kny) = \sum_{m=0}^{\infty} h(x + my) = f(x, y).$$

3. Main results for integrable invariant functions

Suppose that $a(x), b(x)$ and $f(x, y)$ are real functions of x and their derivatives exist. The famous Leibniz's formula states that

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x, t)dt = f(x, b(x))\frac{d}{dx}b(x) - f(x, a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t)dt. \tag{3.1}$$

Theorem 3.1. *Suppose that $f(x, y)$ is an integrable invariant function.*

(i) *We have*

$$\int_{x}^{x+y} f(t, y)dt = \lim_{a\to 0^+} af(x, a).$$

(ii) *We have*

$$f(x + y, y) - f(x, y) = \frac{d}{dx}\lim_{a\to 0^+} af(x, a).$$

If $\frac{\partial f}{\partial x}$ is a piecewise continuous function of x, we also have

$$f(x + y, y) - f(x, y) = \lim_{a\to 0^+} a\frac{\partial f(x, a)}{\partial x}.$$

(iii) *If $\frac{\partial f}{\partial y}$ exists, then*

$$f(x, y) = \int_{x}^{x-y} \frac{\partial}{\partial y} f(t, y)\, dt.$$

(iv) *If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist, setting $g(x, y) = \frac{\partial f(x,y)}{\partial y}$ we have*

$$\frac{\partial f(x, y)}{\partial x} = g(x - y, y) - g(x, y).$$

Proof. Since $\sum_{r=0}^{n-1} f(x + \frac{r}{n}y, y) = f(x, \frac{y}{n})$ for $n \in \mathbb{Z}^+$, we see that

$$\int_0^1 f(x + uy, y)du = \lim_{n\to+\infty} \frac{1}{n}\sum_{r=0}^{n-1} f\big(x + \frac{r}{n}y, y\big) = \lim_{n\to+\infty} \frac{1}{n} f\big(x, \frac{y}{n}\big)$$

and so

$$\int_{x}^{x+y} f(t, y)dt = y\int_0^1 f(x + uy, y)du = \lim_{n\to+\infty} \frac{y}{n} f\big(x, \frac{y}{n}\big) = \lim_{a\to 0^+} af(x, a).$$

This proves part(i). By part(i) and (3.1), we obtain

$$f(x+y,y)-f(x,y)=\frac{\partial}{\partial x}\int_x^{x+y} f(t,y)dt=\frac{d}{dx}\lim_{a\to 0^+} af(x,a).$$

If $\frac{\partial f}{\partial x}$ is a piecewise continuous function of x and $h(x,y)=\frac{\partial f(x,y)}{\partial x}$, we see that

$$f(x+y,y)-f(x,y)=\int_x^{x+y} h(t,y)dt=\lim_{a\to 0^+} ah(x,a).$$

This proves part (ii).

Suppose that $\frac{\partial f}{\partial y}$ exists and $g(x,y)=\frac{\partial}{\partial y}f(x,y)$. By part (i),

$$\frac{\partial}{\partial y}\int_x^{x+y} f(t,y)dt=\frac{d}{dy}\lim_{a\to 0^+} af(x,a)=0.$$

By (3.1),

$$\frac{\partial}{\partial y}\int_x^{x+y} f(t,y)dt=f(x+y,y)+\int_x^{x+y} g(t,y)dt.$$

Thus,

$$f(x+y,y)=-\int_x^{x+y} g(t,y)dt \quad \text{and so} \quad f(x,y)=\int_x^{x-y} g(t,y)\,dt.$$

This proves part (iii).

Now suppose that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and $g(x,y)=\frac{\partial f(x,y)}{\partial y}$. By (3.1) and part (iii), we obtain part (iv).

Summarizing the above proves the theorem.

Theorem 3.2. *Suppose that $f\in I$ and $g(x,y)=\frac{\partial f(x,y)}{\partial y}$.*

(i) *If $g(x,y)$ is an even function of x, then*

$$f(y-x,y)=f(x,y) \quad \textit{and} \quad \int_0^{\frac{y}{2}} f(t,y)dt=\frac{1}{2}\lim_{a\to 0^+} af(0,a).$$

(ii) *If $g(x,y)$ is an odd function of x, then*

$$f(y-x,y)=-f(x,y) \quad \textit{and} \quad \int_0^{y} f(t,y)dt=\lim_{a\to 0^+} af(0,a)=0.$$

Proof. If $g(-x,y)=(-1)^m g(x,y)$, from Theorem 3.1(iii) we see that

$$f(y-x,y)=\int_{y-x}^{-x} g(t,y)dt=\int_x^{x-y} g(-u,y)du=(-1)^m\int_x^{x-y} g(u,y)du=(-1)^m f(x,y).$$

Hence,

$$\int_{\frac{y}{2}}^{y} f(t,y)dt=\int_0^{\frac{y}{2}} f(y-t,y)dt=(-1)^m\int_0^{\frac{y}{2}} f(t,y)dt$$

and so $\int_0^y f(t,y)dt=(1+(-1)^m)\int_0^{\frac{y}{2}} f(t,y)dt$. This together with Theorem 3.1(i) (with $x=0$) yields the result.

Remark 3.1 From Theorems 3.1-3.2 and Examples 2.7, 2.10 and 2.12 (with $y = 1$) one may deduce the following known integrals:

$$\text{(Euler)} \quad \int_0^{\frac{\pi}{2}} \log\sin x dx = -\frac{\pi}{2}\log 2,$$

$$\text{(Poission)} \quad \int_0^{\pi} \log(1 - 2r\text{cos}x + r^2)dx = \begin{cases} 2\pi\log r & \text{if } r > 1, \\ 0 & \text{if } 0 < r < 1, \end{cases}$$

$$\text{(Raabe)} \quad \int_a^{a+1} \log\Gamma(x)dx = a(\log a - 1) + \log\sqrt{2\pi}\ (a > 0).$$

Theorem 3.3. *Suppose that $f \in I$ and $\frac{\partial f}{\partial y}$ exists. For fixed $y > 0$, $f(x, y)$ is a function of x with bounded variation at any closed interval $[a, b]$.*

Proof. Suppose $g(x, y) = \frac{\partial f}{\partial y}$. Then $f(x, y) = \int_x^{x-y} g(t, y)dt$ by Theorem 3.1(iii). For $a = x_0 < x_1 < \cdots < x_n = b$ we see that

$$\begin{aligned}
\sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| &= \sum_{i=0}^{n-1} \Big| \int_{x_{i+1}}^{x_{i+1}-y} g(t, y)dt - \int_{x_i}^{x_i-y} g(t, y)dt \Big| \\
&= \sum_{i=0}^{n-1} \Big| \int_{x_{i+1}}^{x_i} g(t, y)dt + \int_{x_i-y}^{x_{i+1}-y} g(t, y)dt \Big| \\
&\le \sum_{i=0}^{n-1} \Big| \int_{x_i}^{x_{i+1}} g(t, y)dt \Big| + \sum_{i=0}^{n-1} \Big| \int_{x_i-y}^{x_{i+1}-y} g(t, y)dt \Big| \\
&\le \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |g(t, y)|dt + \sum_{i=0}^{n-1} \int_{x_i-y}^{x_{i+1}-y} |g(t, y)|dt \\
&= \int_a^b |g(t, y)|dt + \int_{a-y}^{b-y} |g(t, y)|dt.
\end{aligned}$$

This proves the theorem.

Theorem 3.4 (Product Theorem) . *For $g, h \in I$ define*

$$g * h(x, y) = \int_0^x g(t, y)h(x - t, y)dt + \int_x^y g(t, y)h(x + y - t, y)dt.$$

Then $g * h \in I$ and

$$\int_0^y g * h(x, y)dx = \int_0^y g(x, y)dx \cdot \int_0^y h(x, y)dx.$$

Proof. Set $f(x, y) = g * h(x, y)$, $h_1(x, y) = h(y - x, y)$ and

$$f_1(x, y) = \int_0^y h_1\Big(y\{\frac{t + x}{y}\}, y\Big)g(t, y)dt.$$

Since $h(x, y) \in I$, we have $h_1(x, y) \in I$ and $h_1(y\{\frac{t+x}{y}\}, y) \in I$ by Propositions 2.5 and 2.6. Hence, for $n \in \mathbb{Z}^+$ we have

$$\sum_{r=0}^{n-1} f_1(x + ry, ny) = \sum_{r=0}^{n-1} \int_0^{ny} h_1\Big(ny\{\frac{t + x + ry}{ny}\}, ny\Big)g(t, ny)dt$$

$$\begin{aligned}
&= \int_0^{ny} h_1\Big(y\{\frac{t+x}{y}\},y\Big)g(t,ny)dt \\
&= \sum_{r=0}^{n-1}\int_{ry}^{(r+1)y} h_1\Big(y\{\frac{t+x}{y}\},y\Big)g(t,ny)dt \\
&= \sum_{r=0}^{n-1}\int_0^{y} h_1\Big(y\{\frac{u+ry+x}{y}\},y\Big)g(u+ry,ny)du \\
&= \int_0^{y} h_1\Big(y\{\frac{u+x}{y}\},y\Big)\sum_{r=0}^{n-1} g(u+ry,ny)du \\
&= \int_0^{y} h_1\Big(y\{\frac{u+x}{y}\},y\Big)g(u,y)du = f_1(x,y).
\end{aligned}$$

Thus $f_1 \in I$. For $0 \le x \le y$ we see that

$$\begin{aligned}
f_1(x,y) &= \int_0^{y} h_1\big(y\{\frac{t+x}{y}\},y\big)g(t,y)dt \\
&= \int_0^{y-x} h_1\big(y\cdot\frac{t+x}{y},y\big)g(t,y)dt + \int_{y-x}^{y} h_1\big(y\big(\frac{t+x}{y}-1\big),y\big)g(t,y)dt \\
&= \int_0^{y-x} h_1(t+x,y)g(t,y)dt + \int_{y-x}^{y} h_1(t+x-y,y)g(t,y)dt \\
&= \int_0^{y-x} h(y-x-t,y)g(t,y)dt + \int_{y-x}^{y} h(2y-x-t)g(t,y)dt \\
&= g*h(y-x,y) = f(y-x,y)
\end{aligned}$$

and so $f(x,y) = f_1(y-x,y)$. Hence, for $n \in \mathbb{Z}^+$ and $0 \le x \le y$,

$$\sum_{r=0}^{n-1} f(x+ry,ny) = \sum_{r=0}^{n-1} f_1(y-x+(n-1-r)y,ny) = f_1(y-x,y) = f(x,y).$$

Set $\bar{F}(x,y) = F(x+y,y) - F(x,y)$. Then

$$\begin{aligned}
\bar{f}(x,y) &= f(x+y,y) - f(x,y) \\
&= \int_0^{x+y} g(t,y)h(x+y-t,y)dt + \int_{x+y}^{y} g(t,y)h(x+2y-t,y)dt \\
&\quad - \int_0^{x} g(t,y)h(x-t,y)dt - \int_x^{y} g(t,y)h(x+y-t,y)dt \\
&= \int_0^{x} g(t,y)(h(x-t+y,y) - h(x-t,y))dt \\
&\quad + \int_y^{x+y} (h(x+y-t,y) - h(x+2y-t,y))g(t,y)dt \\
&= \int_0^{x} g(t,y)\bar{h}(x-t,y)dt + \int_0^{x} (h(x-t,y) - h(x+y-t,y))g(t+y,y)dt \\
&= -\int_0^{x} \bar{h}(x-t,y)\bar{g}(t,y)dt.
\end{aligned}$$

For $F(x,y) \in I$ and $n \in \mathbb{Z}^+$, we see that

$$\begin{aligned}\bar{F}(x,ny) &= F(x+ny,ny) - F(x,ny) = \sum_{r=0}^{n-1} F(x+y+ry,ny) - \sum_{r=0}^{n-1} F(x+ry,ny) \\ &= F(x+y,y) - F(x,y) = \bar{F}(x,y).\end{aligned}$$

Hence, for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$,

$$\bar{f}(x,ny) = -\int_0^x \bar{h}(x-t,ny)\bar{g}(t,ny)dt = -\int_0^x \bar{h}(x-t,y)\bar{g}(t,y)dt = \bar{f}(x,y). \tag{3.2}$$

Therefore, if $\sum_{r=0}^{n-1} f(x+ry,ny) = f(x,y)$, then

$$\begin{aligned}\sum_{r=0}^{n-1} f(x+y+ry,ny) &= \sum_{r=0}^{n-1} f(x+ry,ny) + f(x+ny,ny) - f(x,ny) \\ &= f(x,y) + f(x+y,y) - f(x,y) = f(x+y,y)\end{aligned}$$

and

$$\begin{aligned}\sum_{r=0}^{n-1} f(x-y+ry,ny) &= \sum_{r=0}^{n-1} f(x+ry,ny) - f(x-y+ny,ny) + f(x-y,ny) \\ &= f(x,y) - (f(x-y+ny,ny) - f(x-y,ny)) \\ &= f(x,y) - (f(x-y+y,y) - f(x-y,y)) = f(x-y,y).\end{aligned}$$

Since we have proved that $\sum_{r=0}^{n-1} f(x+ry,ny) = f(x,y)$ for $0 \le x \le y$, we must have $\sum_{r=0}^{n-1} f(x+ry,ny) = f(x,y)$ for any $x \in \mathbb{R}$ and $y > 0$. Hence $f(x,y) \in I$.

For fixed $y > 0$ let $H(x,y)$ be a primitive function of $h(x,y)$. That is, $\frac{\partial H}{\partial x} = h$. Set

$$g * H(x,y) = \int_0^x H(x-t,y)g(t,y)dt + \int_x^y H(x+y-t,y)g(t,y)dt.$$

Then

$$g * H(y,y) = \int_0^y H(y-t,y)g(t,y)dt = g * H(0,y).$$

Appealing to (3.1) we see that

$$\begin{aligned}\frac{\partial}{\partial x} g * H(x,y) &= \int_0^x h(x-t,y)g(t,y)dt + \int_x^y h(x+y-t,y)g(t,y)dt \\ &\quad + H(0,y)g(x,y) - H(y,y)g(x,y) \\ &= g * h(x,y) - g(x,y)\int_0^y h(x,y)dx\end{aligned}$$

and so

$$\begin{aligned}0 = g * H(y,y) - g * H(0,y) &= \int_0^y \frac{\partial}{\partial x} g * H(x,y)dx \\ &= \int_0^y g * h(x,y)dx - \int_0^y h(x,y)dx \int_0^y g(x,y)dx.\end{aligned}$$

This completes the proof.

Conjecture 3.1 Suppose $f, g, h \in I$. Then $(f * g) * h = f * (g * h)$.

Theorem 3.5. *Suppose $f \in I$ and*

$$F(x,y) = \int_y^x f(t,y)dt + \frac{1}{y}\int_0^y tf(t,y)dt.$$

Then $F \in I$ and $\frac{\partial}{\partial x}F(x,y) = f(x,y)$.

Proof. By Example 2.2 (with $m = 1$), $\frac{x}{y} - \frac{1}{2} \in I$. Set

$$G(x,y) = \int_0^x \Big(\frac{x-t}{y} - \frac{1}{2}\Big)f(t,y)dt + \int_x^y \Big(\frac{x+y-t}{y} - \frac{1}{2}\Big)f(t,y)dt.$$

Then $G \in I$ by Theorem 3.4. It is clear that

$$G(x,y) = \int_0^y \Big(\frac{x-t}{y} - \frac{1}{2}\Big)f(t,y)dt + \int_x^y f(t,y)dt = \Big(\frac{x}{y} - \frac{1}{2}\Big)\int_0^y f(t,y)dt - F(x,y).$$

By Theorem 3.1, $\int_0^y f(t,y)dt$ is a constant and so $(\frac{x}{y} - \frac{1}{2})\int_0^y f(t,y)dt \in I$. Hence $F(x,y) \in I$. By Leibniz's formula (3.1), $\frac{\partial}{\partial x}F(x,y) = f(x,y)$. Thus the theorem is proved.

For $m, n \in \mathbb{Z}^+$, from [7, pp.26-27] we have

$$B_{2m+1} = 0, \quad B_m(1-t) = (-1)^m B_m(t), \quad \frac{d}{dx}B_m(x) = mB_{m-1}(x), \tag{3.3}$$

$$\int_0^1 B_n(t)dt = \frac{B_{n+1}(1) - B_{n+1}}{n+1} = ((-1)^{n+1} - 1)\frac{B_{n+1}}{n+1} = 0, \tag{3.4}$$

$$\int_0^1 B_m(t)B_n(t)dt = (-1)^{m-1}\frac{B_{m+n}}{\binom{m+n}{m}}. \tag{3.5}$$

For the generalization of (3.5) see [4]. Now, clearly

$$\int_0^1 B_m(1-t)B_n(t)dt = (-1)^m\int_0^1 B_m(t)B_n(t)dt = -\frac{B_{m+n}}{\binom{m+n}{m}} = -\frac{B_{m+n}(1)}{\binom{m+n}{m}}. \tag{3.6}$$

Theorem 3.6. *For any positive integers m and n we have*

$$B_{m+n}(x) = -\binom{m+n}{m}\Big(\int_0^1 B_m(x-t)B_n(t)dt + m\int_x^1 (x-t)^{m-1}B_n(t)dt\Big) \tag{3.7}$$

and

$$\frac{-y^{m-1}B_m(\frac{x}{y})}{m!} * \frac{-y^{n-1}B_n(\frac{x}{y})}{n!} = \frac{-y^{m+n-1}B_{m+n}(\frac{x}{y})}{(m+n)!}.$$

Proof. We first prove (3.7) by induction on m. Since $B_1(x) = x - \frac{1}{2}$, using (3.1), (3.3) and (3.4) we see that

$$\frac{d}{dx}\Big(\int_0^1 B_1(x-t)B_n(t)dt + \int_x^1 B_n(t)dt\Big)$$

$$= \int_0^1 B_n(t)dt - B_n(x) = -B_n(x) = \frac{d}{dx}\Big(- \frac{B_{n+1}(x)}{n+1}\Big).$$

By (3.6), $\int_0^1 B_1(1-t)B_n(t)dt = -\frac{B_{n+1}(1)}{n+1}$. Therefore,

$$\int_0^1 B_1(x-t)B_n(t)dt + \int_x^1 B_n(t)dt = -\frac{B_{n+1}(x)}{n+1}.$$

This shows that (3.7) is true for $m = 1$.

Now assume that (3.7) holds for $m = k$ ($k \in \mathbb{Z}^+$). Appealing to (3.1) and (3.3), we see that

$$\begin{aligned}
&\frac{d}{dx}\Big(\int_0^1 B_{k+1}(x-t)B_n(t)dt + (k+1)\int_x^1 (x-t)^k B_n(t)dt \Big) \\
&= \int_0^1 (k+1)B_k(x-t)B_n(t)dt + (k+1)k\int_x^1 (x-t)^{k-1}B_n(t)dt \\
&= -\frac{k+1}{\binom{k+n}{k}} B_{k+n}(x) = \frac{d}{dx}\Big(- \frac{B_{k+1+n}(x)}{\binom{k+1+n}{k+1}}\Big).
\end{aligned}$$

By (3.6),

$$\int_0^1 B_{k+1}(1-t)B_n(t)dt + (k+1)\int_1^1 (1-t)^k B_n(t)dt = -\frac{B_{k+1+n}(1)}{\binom{k+1+n}{k+1}}.$$

Therefore,

$$-\frac{B_{k+1+n}(x)}{\binom{k+1+n}{k+1}} = \int_0^1 B_{k+1}(x-t)B_n(t)dt + (k+1)\int_x^1 (x-t)^k B_n(t)dt.$$

This shows that (3.7) is true for $m = k+1$. Hence (3.7) is proved by induction.

From [7] we know that $B_m(x+1) = B_m(x) + mx^{m-1}$. Thus, using (3.7) we deduce that

$$\begin{aligned}
&- y^{m-1}\frac{B_m(\frac{x}{y})}{m!} * \Big(- y^{n-1}\frac{B_n(\frac{x}{y})}{n!}\Big) \\
&= \int_0^x y^{n-1}\frac{B_n(\frac{t}{y})}{n!} \cdot y^{m-1}\frac{B_m(\frac{x-t}{y})}{m!}dt + \int_x^y y^{n-1}\frac{B_n(\frac{t}{y})}{n!} \cdot y^{m-1}\frac{B_m(\frac{x+y-t}{y})}{m!}dt \\
&= \frac{y^{m+n-2}}{m!n!}\Big(\int_0^y B_n\Big(\frac{t}{y}\Big)B_m\Big(\frac{x-t}{y}\Big)dt + \int_x^y B_n\Big(\frac{t}{y}\Big)\Big(B_m\Big(\frac{x+y-t}{y}\Big) - B_m\Big(\frac{x-t}{y}\Big)\Big)dt\Big) \\
&= \frac{y^{m+n-2}}{m!n!}\Big(\int_0^y B_m\Big(\frac{x-t}{y}\Big)B_n\Big(\frac{t}{y}\Big)dt + \int_x^y m\Big(\frac{x-t}{y}\Big)^{m-1} B_n\Big(\frac{t}{y}\Big)dt\Big) \\
&= \frac{y^{m+n-1}}{m!n!}\Big(\int_0^1 B_m\Big(\frac{x}{y} - u\Big)B_n(u)du + \int_{\frac{x}{y}}^1 m\Big(\frac{x}{y} - u\Big)^{m-1} B_n(u)du\Big) \\
&= \frac{y^{m+n-1}}{m!n!}\Big(- \frac{B_{m+n}(\frac{x}{y})}{\binom{m+n}{m}}\Big) = -y^{m+n-1}\frac{B_{m+n}(\frac{x}{y})}{(m+n)!}.
\end{aligned}$$

This completes the proof.

Remark 3.2 Theorem 3.6 implies that for $m_1, \cdots, m_k \in \mathbb{Z}^+$,

$$\frac{-y^{m_1-1}B_{m_1}(\frac{x}{y})}{m_1!} * \cdots * \frac{-y^{m_k-1}B_{m_k}(\frac{x}{y})}{m_k!} = \frac{-y^{m_1+\cdots+m_k-1}B_{m_1+\cdots+m_k}(\frac{x}{y})}{(m_1+\cdots+m_k)!}. \tag{3.8}$$

For $\alpha, \beta > 1$, from (2.4) one may deduce that

$$\frac{\zeta(1-\alpha, \frac{x}{y})y^{\alpha-1}}{\Gamma(\alpha)} * \frac{\zeta(1-\beta, \frac{x}{y})y^{\beta-1}}{\Gamma(\beta)} = \frac{\zeta(1-\alpha-\beta, \frac{x}{y})y^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}. \tag{3.9}$$

This can be viewed as a generalization of Theorem 3.6. By (2.4) and Euler's formula $\mathrm{e}^{it} = \cos t + i \sin t$, for $s > 1$ and $0 < x \le y$,

$$y^{s-1}\zeta\Big(1-s, \frac{x}{y}\Big) = \frac{\Gamma(s)}{(2\pi)^s y}\Big(\mathrm{e}^{-\frac{is\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-s}\mathrm{e}^{2\pi i n\frac{x}{y}} + \mathrm{e}^{\frac{is\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-s}\mathrm{e}^{-2\pi i n\frac{x}{y}}\Big). \tag{3.10}$$

Suppose $\alpha, \beta > 1$ and

$$\begin{aligned} f(x,y) = \frac{\zeta(1-\alpha, \frac{x}{y})y^{\alpha-1}}{\Gamma(\alpha)} * \frac{\zeta(1-\beta, \frac{x}{y})y^{\beta-1}}{\Gamma(\beta)} &= \int_0^x \frac{y^{\alpha-1}\zeta(1-\alpha, \frac{x-t}{y})}{\Gamma(\alpha)} \cdot \frac{y^{\beta-1}\zeta(1-\beta, \frac{t}{y})}{\Gamma(\beta)}dt \\ &+ \int_x^y \frac{y^{\alpha-1}\zeta(1-\alpha, \frac{x+y-t}{y})}{\Gamma(\alpha)} \cdot \frac{y^{\beta-1}\zeta(1-\beta, \frac{t}{y})}{\Gamma(\beta)}dt. \end{aligned}$$

By the proof of Theorem 3.4, (2.5) and [1, (1.1.13)],

$$\begin{aligned} &f(x+y,y) - f(x,y) \\ &= -\int_0^x \frac{y^{\alpha-1}}{\Gamma(\alpha)}\Big(\zeta\Big(1-\alpha, 1+\frac{x-t}{y}\Big) - \zeta\Big(1-\alpha, \frac{x-t}{y}\Big)\Big) \\ &\quad \times \frac{y^{\beta-1}}{\Gamma(\beta)}\Big(\zeta\Big(1-\beta, 1+\frac{t}{y}\Big) - \zeta\Big(1-\beta, \frac{t}{y}\Big)\Big)dt \\ &= -\int_0^x \frac{y^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{x-t}{y}\Big)^{\alpha-1} \cdot \frac{y^{\beta-1}}{\Gamma(\beta)}\Big(\frac{t}{y}\Big)^{\beta-1}dt \\ &= -\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{t^{\beta-1}}{\Gamma(\beta)}dt = -\frac{x^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du \\ &= -\frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = \frac{y^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\Big(\zeta\Big(1-\alpha-\beta, 1+\frac{x}{y}\Big) - \zeta\Big(1-\alpha-\beta, \frac{x}{y}\Big)\Big). \end{aligned}$$

On the other hand, for $0 < x \le y$, in view of (3.10) we have

$$\begin{aligned} f(x,y) = \frac{1}{(2\pi)^{\alpha+\beta}y^2}\Big(&\int_0^x \Big(\mathrm{e}^{-\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{2\pi i m\frac{x-t}{y}} + \mathrm{e}^{\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{-2\pi i m\frac{x-t}{y}}\Big) \\ &\times \Big(\mathrm{e}^{-\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{2\pi i n\frac{t}{y}} + \mathrm{e}^{\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{-2\pi i n\frac{t}{y}}\Big)dt \\ &+ \int_x^y \Big(\mathrm{e}^{-\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{2\pi i m\frac{x+y-t}{y}} + \mathrm{e}^{\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{-2\pi i m\frac{x+y-t}{y}}\Big) \end{aligned}$$

$$\times\Big(\mathrm{e}^{-\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{2\pi in\frac{t}{y}}+\mathrm{e}^{\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{-2\pi in\frac{t}{y}}\Big)dt\Big)$$

$$=\frac{1}{(2\pi)^{\alpha+\beta}y^2}\int_0^y\Big(\mathrm{e}^{-\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{2\pi im\frac{x-t}{y}}+\mathrm{e}^{\frac{i\alpha\pi}{2}}\sum_{m=1}^{\infty}\big(\frac{m}{y}\big)^{-\alpha}\mathrm{e}^{-2\pi im\frac{x-t}{y}}\Big)$$

$$\times\Big(\mathrm{e}^{-\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{2\pi in\frac{t}{y}}+\mathrm{e}^{\frac{i\beta\pi}{2}}\sum_{n=1}^{\infty}\big(\frac{n}{y}\big)^{-\beta}\mathrm{e}^{-2\pi in\frac{t}{y}}\Big)dt$$

$$=\frac{1}{(2\pi)^{\alpha+\beta}y^2}\Big(\mathrm{e}^{-\frac{\alpha+\beta}{2}i\pi}I_1+\mathrm{e}^{\frac{\alpha+\beta}{2}i\pi}I_2+\mathrm{e}^{-\frac{\alpha-\beta}{2}i\pi}I_3+\mathrm{e}^{\frac{\alpha-\beta}{2}i\pi}I_4\Big),$$

where

$$I_1=\int_0^y\Big(\sum_{m=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\mathrm{e}^{2\pi im\frac{x-t}{y}}\Big)\Big(\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{2\pi in\frac{t}{y}}\Big)dt,$$

$$I_2=\int_0^y\Big(\sum_{m=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\mathrm{e}^{-2\pi im\frac{x-t}{y}}\Big)\Big(\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{-2\pi in\frac{t}{y}}\Big)dt,$$

$$I_3=\int_0^y\Big(\sum_{m=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\mathrm{e}^{2\pi im\frac{x-t}{y}}\Big)\Big(\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{-2\pi in\frac{t}{y}}\Big)dt,$$

$$I_4=\int_0^y\Big(\sum_{m=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\mathrm{e}^{-2\pi im\frac{x-t}{y}}\Big)\Big(\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{2\pi in\frac{t}{y}}\Big)dt.$$

For $k\in\mathbb{Z}$ it is clear that

$$\int_0^y\mathrm{e}^{2\pi ik\frac{t}{y}}\,dt=\begin{cases}0 & \text{if } k\neq 0,\\ y & \text{if } k=0.\end{cases}$$

Thus,

$$I_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{2\pi im\frac{x}{y}}\int_0^y\mathrm{e}^{2\pi i(n-m)\frac{t}{y}}\,dt=y\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-(\alpha+\beta)}\mathrm{e}^{2\pi in\frac{x}{y}},$$

$$I_2=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{-2\pi im\frac{x}{y}}\int_0^y\mathrm{e}^{2\pi i(m-n)\frac{t}{y}}\,dt=y\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-(\alpha+\beta)}\mathrm{e}^{-2\pi in\frac{x}{y}},$$

$$I_3=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{2\pi im\frac{x}{y}}\int_0^y\mathrm{e}^{-2\pi i(m+n)\frac{t}{y}}\,dt=0,$$

$$I_4=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\frac{m}{y}\Big)^{-\alpha}\Big(\frac{n}{y}\Big)^{-\beta}\mathrm{e}^{-2\pi im\frac{x}{y}}\int_0^y\mathrm{e}^{2\pi i(m+n)\frac{t}{y}}\,dt=0.$$

Therefore, by (3.10),

$$f(x,y)=\frac{1}{(2\pi)^{\alpha+\beta}y}\Big(\mathrm{e}^{-\frac{\alpha+\beta}{2}i\pi}\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-(\alpha+\beta)}\mathrm{e}^{2\pi in\frac{x}{y}}+\mathrm{e}^{\frac{\alpha+\beta}{2}i\pi}\sum_{n=1}^{\infty}\Big(\frac{n}{y}\Big)^{-(\alpha+\beta)}\mathrm{e}^{-2\pi in\frac{x}{y}}\Big)$$

$$=\frac{y^{\alpha+\beta-1}\zeta(1-\alpha-\beta,\frac{x}{y})}{\Gamma(\alpha+\beta)}.$$

This shows that (3.9) holds for $0<x\leq y$. Applying the previous formula for $f(x+y,y)-f(x,y)$, we deduce that (3.9) holds for all $x\in\mathbb{R}$.

Clearly (3.9) implies that for $m_1, \ldots, m_k > 1$,

$$\frac{\zeta(1-m_1,\frac{x}{y})y^{m_1-1}}{\Gamma(m_1)} * \frac{\zeta(1-m_2,\frac{x}{y})y^{m_2-1}}{\Gamma(m_2)} * \cdots * \frac{\zeta(1-m_k,\frac{x}{y})y^{m_k-1}}{\Gamma(m_k)}$$
$$= \frac{\zeta(1-m_1-m_2-\cdots-m_k,\frac{x}{y})y^{m_1+m_2+\cdots+m_k-1}}{\Gamma(m_1+m_2+\cdots+m_k)}$$

and so (3.8) holds for $m_1, \ldots, m_k > 1$ by (2.6).

Declaration of competing interest

The author declares no conflicts of interest regarding the publication of this paper.

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