Congruences for sequences similar to Euler numbers

Zhi-Hong Sun
School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, People's Republic of China

Received 21 July 2010
Revised 5 August 2011
Communicated by David Goss

Abstract

For $a\ne 0$ we define $\{E_n^{(a)}\}$ by $\sum_{k=0}^{[n/2]}\binom{n}{2k}a^{2k}E_{n-2k}^{(a)}=(1-a)^n\quad(n=0,1,2,\ldots)$, where $[n/2]=n/2$ or $(n-1)/2$ according as $2\mid n$ or $2\nmid n$. In the paper we establish many congruences for $E_n^{(a)}$ modulo prime powers, and show that there is a set X and a map $T:\ X\to X$ such that $(-1)^nE_{2n}^{(a)}$ is the number of fixed points of T^n.

MSC: Primary 11B68, Secondary 11A07
Keywords: Euler polynomial, sequence, congruence, p-regular function

1. Introduction

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$(1.1)\qquad \frac{2e^t}{e^{2t}+1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}\ (|t|<\frac{\pi}{2})\quad\text{and}\quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\ (|t|<\pi),$$

E-mail address: zhihongsun@yahoo.com
URL: http://www.hytc.edu.cn/xsjl/szh
The author is supported by the National Natural Sciences Foundation of China (No. 10971078).

which are equivalent to (see [6])

$$E_0=1,\ E_{2n-1}=0,\ \sum_{r=0}^{n}\binom{2n}{2r}E_{2r}=0\ (n\ge 1) \tag{1.2}$$

and

$$E_n(x)+\sum_{r=0}^{n}\binom{n}{r}E_r(x)=2x^n\ (n\ge 0).$$

Euler numbers $\{E_n\}$ is an important sequence of integers and it has many properties and applications. For example, according to [3] we have $E_{(p-1)/2}\equiv 2h(-4p)\pmod p$, where p is a prime of the form $4k+1$ and $h(d)$ is the class number of the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d. In 2005, Arias de Reyna[1] showed that there is a set X and a map $T:\ X\to X$ such that $(-1)^nE_{2n}$ is the number of fixed points of T^n.

In [12] the author introduced the sequence $S_n=4^nE_n(\frac14)$ and showed that $h(-8p)\equiv S_{\frac{p-1}{2}}\pmod p$ for any odd prime p. In [14] the author systematically studied the sequence $U_{2n}=3^{2n}E_{2n}(\frac13)$. Inspired by the properties of $\{E_n\}$, $\{S_n\}$ and $\{U_{2n}\}$, we try to introduce more sequences of integers similar to Euler numbers. For this purpose, we introduce the sequence $\{E_n^{(a)}\}$ for $a\ne 0$ given by

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}a^{2k}E_{n-2k}^{(a)}=(1-a)^n\quad (n=0,1,2,\ldots),$$

where $[x]$ is the greatest integer not exceeding x. Actually, $E_n^{(a)}=(2a)^nE_n(\frac{1}{2a})$, $E_n^{(1)}=E_n$, $E_n^{(2)}=S_n$ and $\{E_n^{(a)}\}$ is a sequence of integers. In the paper we mainly study the properties of $E_n^{(a)}$. We show that there is a set X and a map $T:\ X\to X$ such that $(-1)^nE_{2n}^{(a)}$ is the number of fixed points of T^n. This generalizes Arias de Reyna's result for Euler numbers.

In Section 2 we establish some congruences for $E_n^{(a)}$ modulo a prime. For example, for a prime $p>3$ we have $E_{(p-1)/2}^{(3)}\equiv 0,2h(-4p)$ or $h(-12p)\pmod p$ according as $p\equiv 5\pmod{12}$, $p\equiv 1\pmod{12}$ or $p\equiv 3\pmod 4$.

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of integers and positive integers, respectively. In Section 3 we establish some general congruences for $E_{2^mk+b}^{(a)}$ modulo 2^n, where $a\in\mathbb{Z}$, $k,m,n\in\mathbb{N}$ and $b\in\{0,1,2,\ldots\}$. For example, we determine $E_{2^mk+b}^{(a)}\pmod{2^{m+4+3t}}$, where t is the nonnegative integer given by $2^t\mid a$ and $2^{t+1}\nmid a$. In the case $a=1$, the congruence was given in [13]. The congruence can be viewed as a generalization of the Stern's congruence ([8,16]) $E_{2^mk+b}\equiv E_b-2^mk\pmod{2^{m+1}}$ for even b.

For $m\in\mathbb{N}$ let $\mathbb{Z}_m$ be the set of rational numbers whose denominator is coprime to m. For a prime p, in [10] the author introduced the notion of p-regular functions.

If $f(k)\in\mathbb{Z}_p$ for $k=0,1,2,\dots$ and $\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)\equiv 0 \pmod{p^n}$ for all $n\in\mathbb{N}$, then f is called a p-regular function. If f and g are p-regular functions, from [10, Theorem 2.3] we know that $f\cdot g$ is also a p-regular function.

Let p be an odd prime, and let b be a nonnegative integer. In Section 4 we show that $f_2(k)=\big(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{k(p-1)+b}\big)E^{(2)}_{k(p-1)+b}$ and $f_3(k)=\big(1-(-1)^{[\frac{p+1}{6}]}\big(\frac{p}{3}\big)^{b+1}\ p^{k(p-1)+b}\big)E^{(3)}_{k(p-1)+b}$ are p-regular functions, where $(\frac{a}{m})$ is the Jacobi symbol. Using the properties of p-regular functions in [10,12], we deduce many congruences for $E_n^{(2)}$ and $E_n^{(3)} \pmod{p^m}$. For example, for $k,m\in\mathbb{N}$ we have $E^{(2)}_{k\varphi(p^m)+b}\equiv(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^b)E_b^{(2)} \pmod{p^m}$, where $\varphi(n)$ is Euler's totient function.

In addition to the above notation, we also use throughout this paper the following notation: $\{x\}$—the fractional part of x, $\mathrm{ord}_p n$—the nonnegative integer α such that $p^\alpha\mid n$ but $p^{\alpha+1}\nmid n$ (that is $p^\alpha\,\|\,n$), $\mu(n)$—the Möbius function.

2. Congruences for $E_n^{(a)}$ modulo a prime

Definition 2.1. *For $a\neq 0$ we define $\{E_n^{(a)}\}$ by*

$$\sum_{k=0}^{[n/2]}\binom{n}{2k}a^{2k}E^{(a)}_{n-2k}=(1-a)^n \quad (n=0,1,2,\dots).$$

By the definition we have $E_n^{(a)}\in\mathbb{Z}$ for $a\in\mathbb{Z}$ and $E_n^{(1)}=E_n$. The first few Euler numbers are shown below:

$$E_0=1,\ E_2=-1,\ E_4=5,\ E_6=-61,\ E_8=1385,\ E_{10}=-50521,$$
$$E_{12}=2702765,\ E_{14}=-199360981,\ E_{16}=19391512145.$$

The first few values of $E_n^{(2)}$ and $E_n^{(3)}$ are given below:

$$E_0^{(2)}=1,\ E_1^{(2)}=-1,\ E_2^{(2)}=-3,\ E_3^{(2)}=11,\ E_4^{(2)}=57,\ E_5^{(2)}=-361,\ E_6^{(2)}=-2763,$$
$$E_7^{(2)}=24611,\ E_8^{(2)}=250737,\ E_9^{(2)}=-2873041,\ E_{10}^{(2)}=-36581523;$$
$$E_0^{(3)}=1,\ E_1^{(3)}=-2,\ E_2^{(3)}=-5,\ E_3^{(3)}=46,\ E_4^{(3)}=205,\ E_5^{(3)}=-3362,$$
$$E_6^{(3)}=-22265,\ E_7^{(3)}=515086,\ E_8^{(3)}=4544185,\ E_9^{(3)}=-135274562.$$

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\ (n\ge 0).$$

It is well known that (see [6])

(2.1)

$$\begin{aligned}E_n(x) &= \frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}(2x-1)^{n-r}E_r\\ &= \frac{2}{n+1}\Big(B_{n+1}(x)-2^{n+1}B_{n+1}\Big(\frac{x}{2}\Big)\Big) = \frac{2^{n+1}}{n+1}\Big(B_{n+1}\Big(\frac{x+1}{2}\Big)-B_{n+1}\Big(\frac{x}{2}\Big)\Big).\end{aligned}$$

In particular,

$$E_n = 2^nE_n\Big(\frac{1}{2}\Big) \quad \text{and} \quad E_n(0) = \frac{2(1-2^{n+1})B_{n+1}}{n+1}. \tag{2.2}$$

It is also known that (see [6])

$$B_{2n+3} = 0,\ B_n(1-x) = (-1)^nB_n(x) \quad \text{and} \quad E_n(1-x) = (-1)^nE_n(x). \tag{2.3}$$

Theorem 2.1. *Let n be a nonnegative integer and $a \neq 0$. Then*

$$\begin{aligned}E_n^{(a)} &= (2a)^nE_n\Big(\frac{1}{2a}\Big) = \sum_{k=0}^{[n/2]}\binom{n}{2k}(1-a)^{n-2k}a^{2k}E_{2k}\\ &= \sum_{k=0}^{n}\binom{n}{k}2^{k+1}(1-2^{k+1})\frac{B_{k+1}}{k+1}a^k.\end{aligned}$$

Proof. By Definition 2.1 we have

$$\begin{aligned}\mathrm{e}^{(1-a)t} &= \sum_{n=0}^{\infty}(1-a)^n\frac{t^n}{n!} = \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{[n/2]}\binom{n}{2k}a^{2k}E_{n-2k}^{(a)}\Big)\frac{t^n}{n!}\\ &= \Big(\sum_{k=0}^{\infty}a^{2k}\frac{t^{2k}}{(2k)!}\Big)\Big(\sum_{m=0}^{\infty}E_m^{(a)}\frac{t^m}{m!}\Big) = \frac{\mathrm{e}^{at}+\mathrm{e}^{-at}}{2}\Big(\sum_{m=0}^{\infty}E_m^{(a)}\frac{t^m}{m!}\Big).\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty}E_n^{(a)}\frac{t^n}{n!} = \frac{\mathrm{e}^{(1-a)t}}{(\mathrm{e}^{at}+\mathrm{e}^{-at})/2} = \frac{2\mathrm{e}^t}{\mathrm{e}^{2at}+1}. \tag{2.4}$$

From (1.1) we know that $\sum_{n=0}^{\infty}E_n(\frac{1}{2a})\frac{(2at)^n}{n!} = \frac{2\mathrm{e}^t}{\mathrm{e}^{2at}+1}$. Hence, from the above and (2.1) we deduce

$$E_n^{(a)} = (2a)^nE_n\Big(\frac{1}{2a}\Big) = \sum_{r=0}^{n}\binom{n}{r}(1-a)^{n-r}a^rE_r = \sum_{k=0}^{[n/2]}\binom{n}{2k}(1-a)^{n-2k}a^{2k}E_{2k}.$$

By (1.1) and (2.2) we have

$$\sum_{n=0}^{\infty}\frac{2(1-2^{n+1})B_{n+1}}{n+1}\cdot\frac{(2at)^n}{n!} = \sum_{n=0}^{\infty}E_n(0)\frac{(2at)^n}{n!} = \frac{2}{\mathrm{e}^{2at}+1}.$$

Thus

$$\begin{aligned}\sum_{n=0}^{\infty} E_n^{(a)} \frac{t^n}{n!} &= \mathrm{e}^t \cdot \frac{2}{\mathrm{e}^{2at}+1} = \Big(\sum_{m=0}^{\infty} \frac{t^m}{m!}\Big)\Big(\sum_{k=0}^{\infty} 2^{k+1}(1-2^{k+1})a^k \frac{B_{k+1}}{k+1}\Big)\frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k}2^{k+1}(1-2^{k+1})a^k\frac{B_{k+1}}{k+1}\Big)\frac{t^n}{n!}\end{aligned}$$

and so $E_n^{(a)} = \sum_{k=0}^{n}\binom{n}{k}2^{k+1}(1-2^{k+1})a^k\frac{B_{k+1}}{k+1}$. The proof is now complete.

Corollary 2.1. *Let $a \neq 0$ and $n \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_k^{(a)} = \begin{cases} 0 & \text{if } 2 \mid n, \\ 2^{n+1}(2^{n+1}-1)a^n \dfrac{B_{n+1}}{n+1} & \text{if } 2 \nmid n. \end{cases}$$

Proof. By Theorem 2.1 and the binomial inversion formula we have $\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_k^{(a)} = 2^{n+1}(1-2^{n+1})a^n\frac{B_{n+1}}{n+1}$. Noting that $B_{n+1}=0$ for even n we deduce the result.

Lemma 2.1. *For $n \in \mathbb{N}$ we have $E_{2n}^{(3)} = \frac{1}{2}(3^{2n}+1)E_{2n}$.*

Proof. Using (1.1), (1.2) and (2.4) we see that

$$\begin{aligned}2\sum_{n=0}^{\infty} E_{2n}^{(3)}\frac{t^{2n}}{(2n)!} &= \sum_{n=0}^{\infty} E_n^{(3)}\frac{t^n+(-t)^n}{n!} = \frac{2\mathrm{e}^t}{\mathrm{e}^{6t}+1} + \frac{2\mathrm{e}^{-t}}{\mathrm{e}^{-6t}+1} = \frac{2\mathrm{e}^t+2\mathrm{e}^{5t}}{\mathrm{e}^{6t}+1} \\ &= \frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1} + \frac{2\mathrm{e}^{3t}}{\mathrm{e}^{6t}+1} = \sum_{n=0}^{\infty}(1+3^n)E_n\frac{t^n}{n!} = \sum_{n=0}^{\infty}(1+3^{2n})E_{2n}\frac{t^{2n}}{(2n)!}.\end{aligned}$$

So the result follows.

In [3], Ernvall showed that for a prime $p \equiv 1 \pmod 4$,

$$E_{(p-1)/2} \equiv 2h(-4p) \pmod p \quad \text{and so} \quad p \nmid E_{(p-1)/2}. \tag{2.5}$$

In [12] the author defined $\{S_n\}$ by $S_n = 1 - \sum_{k=0}^{n-1}\binom{n}{k}2^{2n-2k-1}S_k$ $(n \geq 0)$ and showed that $S_n = 4^n E_n(\frac{1}{4})$. Thus, by Theorem 2.1 we have $S_n = E_n^{(2)}$. From [12, Theorem 3.1 and Corollary 3.1] we know that for any odd prime p,

$$h(-8p) \equiv E_{(p-1)/2}^{(2)} \pmod p \quad \text{and hence} \quad p \nmid E_{(p-1)/2}^{(2)}. \tag{2.6}$$

Now we state the similar congruence for $E_{(p-1)/2}^{(3)} \pmod p$.

Theorem 2.2. *Let p be a prime greater than 3. Then*

$$E_{\frac{p-1}{2}}^{(3)} \equiv \begin{cases} 0 \pmod p & \text{if } p \equiv 5 \pmod{12}, \\ E_{(p-1)/2} \equiv 2h(-4p) \pmod p & \text{if } p \equiv 1 \pmod{12}, \\ h(-12p) \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. If $p \equiv 1 \pmod 4$, by Lemma 2.1 and (2.5) we have

$$\begin{aligned} E^{(3)}_{\frac{p-1}{2}} &= \frac{1}{2}(3^{\frac{p-1}{2}}+1)E_{\frac{p-1}{2}} \equiv \frac{1}{2}\Big(1+\Big(\frac{3}{p}\Big)\Big)E_{\frac{p-1}{2}} \\ &\equiv \begin{cases} 0 \pmod p & \text{if } p \equiv 5 \pmod{12}, \\ E_{\frac{p-1}{2}} \equiv 2h(-4p) \pmod p & \text{if } p \equiv 1 \pmod{12}. \end{cases} \end{aligned}$$

Now assume $p \equiv 3 \pmod 4$. It is known that (see [6]) $B_{2n}\big(\frac{1}{6}\big) = \frac{1}{2}(2^{1-2n}-1)(3^{1-2n}-1)B_{2n}$. Thus,

$$\begin{aligned} B_{\frac{p+1}{2}}\big(\frac{1}{6}\big) &= \frac{1}{2}(2^{-\frac{p-1}{2}}-1)(3^{-\frac{p-1}{2}}-1)B_{\frac{p+1}{2}} \equiv \frac{1}{2}\Big(\Big(\frac{2}{p}\Big)-1\Big)\Big(\Big(\frac{3}{p}\Big)-1\Big)B_{\frac{p+1}{2}} \\ &\equiv \begin{cases} 0 \pmod p & \text{if } p \equiv 7, 11, 23 \pmod{24}, \\ 2B_{\frac{p+1}{2}} \pmod p & \text{if } p \equiv 19 \pmod{24}. \end{cases} \end{aligned}$$

Hence, by Theorem 2.1 and (2.1) we have

$$\begin{aligned} E^{(3)}_{\frac{p-1}{2}} &= 6^{\frac{p-1}{2}}E_{\frac{p-1}{2}}\Big(\frac{1}{6}\Big) = 6^{\frac{p-1}{2}} \cdot \frac{2}{(p+1)/2}\Big(B_{\frac{p+1}{2}}\Big(\frac{1}{6}\Big) - 2^{\frac{p+1}{2}}B_{\frac{p+1}{2}}\Big(\frac{1}{12}\Big)\Big) \\ &\equiv 4\Big(\frac{6}{p}\Big)\Big(B_{\frac{p+1}{2}}\big(\frac{1}{6}\big) - 2\Big(\frac{2}{p}\Big)B_{\frac{p+1}{2}}\big(\frac{1}{12}\big)\Big) \\ &\equiv \begin{cases} 8B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) \pmod p & \text{if } p \equiv 7 \pmod{24}, \\ -8B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) \pmod p & \text{if } p \equiv 11, 23 \pmod{24}, \\ 8B_{\frac{p+1}{2}} + 8B_{\frac{p+1}{2}}\big(\frac{1}{12}\big) \pmod p & \text{if } p \equiv 19 \pmod{24}. \end{cases} \end{aligned}$$

Now applying [12, Theorem 3.2(ii)] we obtain $E^{(3)}_{(p-1)/2} \equiv h(-12p) \pmod p$. So the theorem is proved.

Remark 2.1 In a similar way, one can show that for any prime $p \equiv 11, 19 \pmod{20}$, $(1+2(-1)^{\frac{p+1}{4}})h(-5p) \equiv 2E^{(\frac{5}{2})}_{\frac{p-1}{2}} \pmod p$.

Corollary 2.2. *Let p be an odd prime with $p \not\equiv 5 \pmod{12}$. Then $p \nmid E^{(3)}_{(p-1)/2}$.*

Proof. For $p \equiv 3 \pmod 4$, it is well known that ([15, pp.3-5]) $1 \le h(-12p) < p$. So the result follows from Theorem 2.2 and (2.5).

Theorem 2.3. *Let p be an odd prime, $m \in \{2, 3, 4, \ldots\}$ and $p \equiv \pm 1 \pmod m$. Then*

$$\sum_{\frac{p}{m}<i<\frac{p}{2}} (-1)^{i-1}\frac{1}{i} \equiv \mp(-1)^{[\frac{p}{m}]}\frac{m}{2}E^{(m/2)}_{p-2} \pmod p$$

and

$$\sum_{i=1}^{[p/m]}(-1)^{i-1}\frac{1}{i^k}\equiv-(-1)^{[p/m]}\frac{m^k}{2}E_{p-1-k}^{(m/2)} \pmod p \quad \textit{for } k=2,4,\dots,p-3.$$

Proof. Let $k\in\{1\}\cup\{2,4,\dots,p-3\}$. Putting $s=1$ and substituting k by $p-1-k$ in [12, Corollary 2.2] we see that

$$\begin{aligned}&E_{p-1-k}(0)-(-1)^{[\frac{p}{m}]}E_{p-1-k}\Big(\Big\{\frac{p}{m}\Big\}\Big)\\&\equiv 2(-1)^{p-1-k-1}\sum_{i=1}^{[p/m]}(-1)^i i^{p-1-k}\equiv 2(-1)^k\sum_{i=1}^{[p/m]}(-1)^{i-1}\frac{1}{i^k} \pmod p.\end{aligned}$$

It is well known that ([5]) $pB_{p-1}\equiv p-1 \pmod p$. Thus, in view of (2.2) and (2.3) we have

$$E_{p-1-k}(0)=\frac{2(1-2^{p-k})B_{p-k}}{p-k}=\begin{cases}0 \pmod p & \text{if } 2\mid k,\\ \dfrac{2-2^p}{p}\cdot\dfrac{pB_{p-1}}{p-1}\equiv-\dfrac{2^p-2}{p} \pmod p & \text{if } k=1.\end{cases}$$

Using (2.3) and Theorem 2.1 we see that

$$E_{p-2}\Big(\Big\{\frac{p}{m}\Big\}\Big)=\begin{cases}E_{p-2}(\frac{1}{m})=m^{2-p}E_{p-2}^{(m/2)}\equiv mE_{p-2}^{(m/2)} \pmod p & \text{if } m\mid p-1,\\ E_{p-2}(1-\frac{1}{m})=-E_{p-2}(\frac{1}{m})\equiv-mE_{p-2}^{(m/2)} \pmod p & \text{if } m\mid p+1.\end{cases}$$

From the above we deduce

$$\sum_{i=1}^{[p/m]}(-1)^{i-1}\frac{1}{i}\equiv\frac{2^{p-1}-1}{p}\pm(-1)^{[p/m]}\frac{m}{2}E_{p-2}^{(m/2)} \pmod p.$$

Taking $m=2$ we have the known result $\sum_{i=1}^{[p/2]}(-1)^{i-1}\frac{1}{i}\equiv\frac{2^{p-1}-1}{p} \pmod p$. Hence

$$\sum_{\frac{p}{m}<i<\frac{p}{2}}(-1)^{i-1}\frac{1}{i}=\sum_{i=1}^{[p/2]}(-1)^{i-1}\frac{1}{i}-\sum_{i=1}^{[p/m]}(-1)^{i-1}\frac{1}{i}\equiv\mp(-1)^{[p/m]}\frac{m}{2}E_{p-2}^{(m/2)} \pmod p.$$

For $k\in\{2,4,\dots,p-3\}$, using (2.3) and Theorem 2.1 we see that

$$E_{p-1-k}\Big(\Big\{\frac{p}{m}\Big\}\Big)=E_{p-1-k}\Big(\frac{1}{m}\Big)=m^{-(p-1-k)}E_{p-1-k}^{(m/2)}\equiv m^kE_{p-1-k}^{(m/2)} \pmod p.$$

Now putting all the above together we deduce the result.

3. Congruences for $E_n^{(a)}$ modulo 2^m

In [12] the author established many congruences for $E_{2n} \pmod{2^m}$, where $m,n \in \mathbb{N}$. In the section we extend such congruences to $E_{2n+b}^{(a)} \pmod{2^m}$, where a is a nonzero integer and $b \in \{0,1,2,\dots\}$.

Lemma 3.1. *Let s and n be nonnegative integers. Then*

(i)

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k}{s} = \begin{cases} \binom{n}{s-n}2^{2n-s} & \textit{if } s \ge n, \\ 0 & \textit{if } s < n. \end{cases}$$

(ii)

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k+1}{s} = \begin{cases} \frac{s+1}{n+1}\binom{n+1}{s-n}2^{2n-s} & \textit{if } s \ge n, \\ 0 & \textit{if } s < n. \end{cases}$$

Proof. (i) can be found in [4, (3.64)]. We now use (i) to deduce (ii). By (i) we have

$$\begin{aligned} &\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k+1}{s} \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\binom{2k}{s}+\binom{2k}{s-1}\right) \\ &= \begin{cases} \binom{n}{s-n}2^{2n-s}+\binom{n}{s-1-n}2^{2n-(s-1)} = \frac{s+1}{n+1}\binom{n+1}{s-n}2^{2n-s} & \text{if } s \ge n+1, \\ 2^n+0 & \text{if } s = n, \\ 0+0 & \text{if } s < n. \end{cases} \end{aligned}$$

This proves (ii).

Theorem 3.1.*Let a be a nonzero integer, $n \in \mathbb{N}$ and let b be a nonnegative integer. Suppose that $\alpha_n \in \mathbb{N}$ is given by $2^{\alpha_n - 1} \le n < 2^{\alpha_n}$.*

(i) *If p is an odd prime divisor of a, then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}^{(a)} \equiv \begin{cases} 0 \pmod{p^{n\,\mathrm{ord}_p a}} & \textit{if } 2 \nmid n, \\ 0 \pmod{p^{(n+1)\mathrm{ord}_p a}} & \textit{if } 2 \mid n. \end{cases}$$

(ii) *We have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k}^{(a)} \equiv \begin{cases} 0 \pmod{2^{(n+1)\mathrm{ord}_2 a-\alpha_n+\mathrm{ord}_2 n+2n}} & \textit{if } 2 \mid n, \\ 0 \pmod{2^{n\,\mathrm{ord}_2 a+2n-\alpha_n}} & \textit{if } 2 \nmid n \end{cases}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k+1}^{(a)} \equiv \begin{cases} 0 \pmod{2^{(n+1)\mathrm{ord}_2 a+2n}} & \text{if } 2 \mid n, \\ 0 \pmod{2^{n\mathrm{ord}_2 a+2n-\mathrm{ord}_2(n+1)}} & \text{if } 2 \nmid n. \end{cases}$$

(iii) *We have*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}E_{2k+b}^{(a)} \equiv 0 \pmod{2^{(2+\mathrm{ord}_2 a)n-\alpha_n}}.$$

Moreover, if $2 \mid n$ *and* $2 \nmid b$, *then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}E_{2k+b}^{(a)} \equiv 0 \pmod{2^{(2+\mathrm{ord}_2 a)(n+1)-\alpha_n}}.$$

Proof. Using Theorem 2.1, Lemma 3.1 and (2.3) we see that

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k}^{(a)} &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{s=0}^{2k}\binom{2k}{s}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1} \\
&= \sum_{s=0}^{2n}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1}\sum_{\frac{s}{2}\le k\le n}\binom{n}{k}(-1)^{n-k}\binom{2k}{s} \\
&= \sum_{s=1}^{2n}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k}{s} \\
&= \sum_{s=n}^{2n}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1}\binom{n}{s-n}2^{2n-s} \\
&= \sum_{\substack{s=n \\ 2\nmid s}}^{2n-1}\frac{2^{2n+1}a^s(1-2^{s+1})B_{s+1}}{s+1}\binom{n}{s-n}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}E_{2k+1}^{(a)} &= \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\sum_{s=0}^{2k+1}\binom{2k+1}{s}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1} \\
&= \sum_{s=0}^{2n+1}\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\binom{2k+1}{s} \\
&= \frac{1}{n+1}\sum_{\substack{s=n \\ 2\nmid s}}^{2n+1}2^{2n+1}a^s(1-2^{s+1})B_{s+1}\binom{n+1}{s-n}.
\end{aligned}$$

From Corollary 2.1 we see that for odd s,

$$\frac{2^{s+1}(1-2^{s+1})B_{s+1}}{s+1} = -\sum_{r=0}^{s}\binom{s}{r}(-1)^r E_r \in \mathbb{Z}. \tag{3.1}$$

Thus, if p is an odd prime with $p \mid a$, then $\frac{2^{s+1}a^s(1-2^{s+1})B_{s+1}}{s+1} \equiv 0 \pmod{p^{s\,\mathrm{ord}_p a}}$. Now, from the above we deduce that for $i = 0, 1$,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+i}^{(a)} \equiv \begin{cases} 0 \pmod{p^{n\,\mathrm{ord}_p a}} & \text{if } 2 \nmid n, \\ 0 \pmod{p^{(n+1)\mathrm{ord}_p a}} & \text{if } 2 \mid n. \end{cases} \tag{3.2}$$

From [10, (2.5)] we know that for any function f,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k f(k+m) = \sum_{k=0}^{m} \binom{m}{k} (-1)^k \sum_{r=0}^{k+n} \binom{k+n}{r} (-1)^r f(r). \tag{3.3}$$

Thus,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b}^{(a)} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2(k+[\frac{b}{2}])+b-2[\frac{b}{2}]}^{(a)} \\ &= \sum_{k=0}^{[b/2]} \binom{[\frac{b}{2}]}{k} (-1)^k \sum_{r=0}^{k+n} \binom{k+n}{r} (-1)^r E_{2r+b-2[\frac{b}{2}]}^{(a)}. \end{aligned} \tag{3.4}$$

Now applying (3.2) we deduce (i).

Suppose $s \in \{n, n+1, \ldots, 2n-1\}$ and $2 \nmid s$. If $2 \mid n$, then $2 \nmid s-n$ and so $\binom{n}{s-n} = \frac{n}{s-n}\binom{n-1}{s-n-1} \equiv 0 \pmod{2^{\mathrm{ord}_2 n}}$. Since $2B_{s+1} \equiv 1 \pmod 2$ and $2^{\mathrm{ord}_2(s+1)} \le s+1 \le 2n < 2^{\alpha_n+1}$, we see that $\mathrm{ord}_2(s+1) \le \alpha_n$ and so

$$\begin{aligned} \mathrm{ord}_2\Big(\frac{2a^s(1-2^{s+1})B_{s+1}}{s+1} \binom{n}{s-n}\Big) &\ge s\,\mathrm{ord}_2 a - \alpha_n + \mathrm{ord}_2 n \\ &\ge (n+1)\mathrm{ord}_2 a - \alpha_n + \mathrm{ord}_2 n. \end{aligned}$$

If $2 \nmid n$, we see that

$$\mathrm{ord}_2\Big(\frac{2a^s(1-2^{s+1})B_{s+1}}{s+1} \binom{n}{s-n}\Big) \ge \mathrm{ord}_2\Big(\frac{a^s}{s+1}\Big) \ge n\mathrm{ord}_2 a - \alpha_n.$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k}^{(a)} &= \sum_{\substack{s=n \\ 2\nmid s}}^{2n-1} \frac{2a^s(1-2^{s+1})B_{s+1}}{s+1} \binom{n}{s-n} 2^{2n} \\ &\equiv \begin{cases} 0 \pmod{2^{(n+1)\mathrm{ord}_2 a - \alpha_n + \mathrm{ord}_2 n + 2n}} & \text{if } 2 \mid n, \\ 0 \pmod{2^{n\mathrm{ord}_2 a + 2n - \alpha_n}} & \text{if } 2 \nmid n. \end{cases} \end{aligned}$$

Since $2B_{s+1} \equiv 1 \pmod 2$ for odd s we also have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} E_{2k+1}^{(a)} &= \frac{1}{n+1} \sum_{\substack{s=n \\ 2\nmid s}}^{2n+1} 2^{2n+1} a^s (1-2^{s+1}) B_{s+1} \binom{n+1}{s-n} \\ &\equiv \begin{cases} 0 \pmod{2^{(n+1)\mathrm{ord}_2 a + 2n}} & \text{if } 2 \mid n, \\ 0 \pmod{2^{n\mathrm{ord}_2 a + 2n - \mathrm{ord}_2(n+1)}} & \text{if } 2 \nmid n. \end{cases} \end{aligned}$$

So (ii) holds.

Since $2^{\mathrm{ord}_2(n+1)} \le n+1 \le 2^{\alpha_n}$ we see that $\mathrm{ord}_2(n+1) \le \alpha_n$. Thus, from (ii) we deduce

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+i}^{(a)} \equiv 0 \pmod{2^{2n+n\mathrm{ord}_2 a-\alpha_n}} \quad \text{for} \quad i=0,1. \tag{3.5}$$

As $\alpha_{s+1} = \alpha_s$ or α_s+1, we see that $2(s+1)-\alpha_{s+1} \ge 2s-\alpha_s$ and hence $2r-\alpha_r \ge 2s-\alpha_s$ for $r \ge s$. For $k \ge 0$, by (3.5) we have

$$\sum_{r=0}^{k+n} \binom{k+n}{r} (-1)^r E_{2r+b-2[\frac{b}{2}]}^{(a)} \equiv 0 \pmod{2^{2(k+n)-\alpha_{k+n}+(n+k)\mathrm{ord}_2 a}}.$$

Since $2(k+n)-\alpha_{k+n} \ge 2n-\alpha_n$, we must have

$$\sum_{r=0}^{k+n} \binom{k+n}{r} (-1)^r E_{2r+b-2[\frac{b}{2}]}^{(a)} \equiv 0 \pmod{2^{2n-\alpha_n+n\mathrm{ord}_2 a}}.$$

Combining this with (3.4) we obtain

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k E_{2k+b}^{(a)} \equiv 0 \pmod{2^{(2+\mathrm{ord}_2 a)n-\alpha_n}}.$$

Now we assume $2 \mid n$ and $2 \nmid b$. For $k,n \in \mathbb{N}$ we have $n+k \ge 2 > \frac{17}{15}$ and so $\frac{n+k+1}{n+k-1} < 16$. Hence

$$\log_2(n+k+1) - \log_2(n+k-1) = \log_2 \frac{n+k+1}{n+k-1} < 4$$

and so

$$(n+k)(2+\mathrm{ord}_2 a) - \log_2(n+k+1) > (n+k-2)(2+\mathrm{ord}_2 a) - \log_2(n+k-1).$$

Since $2^{\mathrm{ord}_2(n+k+1)} \le n+k+1$ and $n+2 \le 2^{\alpha_n}$ we see that $\mathrm{ord}_2(n+k+1) \le \log_2(n+k+1)$ and $\log_2(n+2) \le \alpha_n$. Thus, for odd k we have

$$\begin{aligned}
(n+k)(2+\mathrm{ord}_2 a) - \mathrm{ord}_2(n+k+1) &\ge (n+k)(2+\mathrm{ord}_2 a) - \log_2(n+k+1) \\
&\ge (n+k-2)(2+\mathrm{ord}_2 a) - \log_2(n+k-1) \\
&\ge \cdots \ge (n+1)(2+\mathrm{ord}_2 a) - \log_2(n+2) \\
&\ge (n+1)(2+\mathrm{ord}_2 a) - \alpha_n
\end{aligned}$$

and so (by (ii))

$$\sum_{r=0}^{n+k} \binom{n+k}{r} (-1)^r E_{2r+1}^{(a)} \equiv 0 \pmod{2^{(n+1)(2+\mathrm{ord}_2 a)-\alpha_n}}. \tag{3.6}$$

For even k, using (ii) and the fact

$$(n+k+1)\mathrm{ord}_2 a+2(n+k)\geq (n+1)\mathrm{ord}_2 a+2n\geq (n+1)(2+\mathrm{ord}_2 a)-\alpha_n$$

we see that (3.6) is also true. Thus applying (3.4) we deduce that $\sum_{k=0}^{n}\binom{n}{k}(-1)^k E_{2k+b}^{(a)}\equiv 0 \pmod{2^{(n+1)(2+\mathrm{ord}_2 a)-\alpha_n}}$. This completes the proof.

Corollary 3.1. *Let a be a nonzero integer and $b\in\{0,1,2,\dots\}$. Then $f(k)=E_{2k+b}^{(a)}$ is a $2-$regular function.*

Proof. Let $\alpha_n\in\mathbb{N}$ be given by $2^{\alpha_n-1}\leq n<2^{\alpha_n}$. As $2^n>n$, we see that $\alpha_n\leq n$ and so $2n-\alpha_n\geq n$. Now applying Theorem 3.1(iii) we obtain the result.

Theorem 3.2. *Suppose that a is a nonzero integer, $k,m,n,t\in\mathbb{N}$ and $b\in\{0,1,2,\dots\}$. For $s\in\mathbb{N}$ let $\alpha_s\in\mathbb{N}$ be given by $2^{\alpha_s-1}\leq s<2^{\alpha_s}$ and let $e_s(a,b)=2^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^r E_{2r+b-2[\frac{b}{2}]}^{(a)}$. Then*

$$E_{2^m kt+b}^{(a)}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}^{(a)} \pmod{2^{mn+(1+\mathrm{ord}_2 a)n-\alpha_n}}.$$

Moreover,

$$\begin{aligned}E_{2^m kt+b}^{(a)}\equiv&\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}^{(a)}\\&+2^{mn}\binom{k}{n}(-t)^n e_n(a,b) \pmod{2^{mn+(1+\mathrm{ord}_2 a)(n+1)-\alpha_{n+1}}}.\end{aligned}$$

In particular, when $2\mid n$ and $2\nmid b$, we have

$$E_{2^m kt+b}^{(a)}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}^{(a)} \pmod{2^{mn+(1+\mathrm{ord}_2 a)(n+1)-\alpha_{n+1}}}.$$

Proof. For $r\in\mathbb{N}$ set $A_r(a,b)=2^{-r}\sum_{s=0}^{r}\binom{r}{s}(-1)^s E_{2s+b}^{(a)}$. As $\alpha_r\leq r$, using Theorem 3.1(iii) we see that $A_r(a,b)\in\mathbb{Z}_2$ and

$$A_r(a,b)\equiv\begin{cases}0 \pmod{2^{(1+\mathrm{ord}_2 a)r-\alpha_r+2+\mathrm{ord}_2 a}} & \text{if } 2\mid r \text{ and } 2\nmid b,\\ 0 \pmod{2^{(1+\mathrm{ord}_2 a)r-\alpha_r}} & \text{otherwise.}\end{cases} \tag{3.7}$$

By [9, Lemma 2.1] we have

$$\begin{aligned}&E_{2^m kt+b}^{(a)}-\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m rt+b}^{(a)}\\&=\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s E_{2^m st+b}^{(a)}.\end{aligned} \tag{3.8}$$

From Corollary 3.1 and the proof of [12, Theorem 4.2] we know that

$$\begin{aligned}&\sum_{s=0}^{n}\binom{n}{s}(-1)^s E^{(a)}_{2^m st+b}\\ &= A_n(a,b)t^n\cdot 2^{mn} + \sum_{r=n+1}^{2^{m-1}nt}(-2)^n(-1)^r A_r(a,b)\Big(\frac{(-1)^{r-n}s(r,n)n!}{r!}2^{r-n}\\ &\quad\times 2^{(m-1)n}t^n + \sum_{j=n+1}^{r}\frac{(-1)^{r-j}s(r,j)j!}{r!}2^{r-j}\cdot\frac{S(j,n)n!}{j!}2^{j-n}\cdot(2^{m-1}t)^j\Big),\end{aligned} \tag{3.9}$$

where $\{s(n,k)\}$ and $\{S(n,k)\}$ are Stirling numbers given by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

By [12, Lemma 4.2], for $n+1\le j\le r$ we have

$$\frac{s(r,j)j!}{r!}2^{r-j},\ \frac{S(j,n)n!}{j!}2^{j-n}\in\mathbb{Z}_2 \text{ and } \frac{s(r,n)n!}{r!}2^{r-n}\equiv\binom{n}{r-n} \pmod 2. \tag{3.10}$$

As $\alpha_{s+1}\le\alpha_s+1$ we have $s+1-\alpha_{s+1}\ge s-\alpha_s$ and hence $r-\alpha_r\ge s-\alpha_s$ for $r\ge s$. Therefore, by (3.7) we have $2^{(n+1)\mathrm{ord}_2 a+n+1-\alpha_{n+1}}\mid A_r(a,b)$ for $r\ge n+1$. Hence, using (3.9) we get

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r E^{(a)}_{2^m rt+b}\equiv 2^{mn}t^nA_n(a,b) \pmod{2^{mn+(n+1)(\mathrm{ord}_2 a+1)-\alpha_{n+1}}}. \tag{3.11}$$

From (3.7) we have $2^{(1+\mathrm{ord}_2 a)n-\alpha_n}\mid A_n(a,b)$. Since $\alpha_{n+1}=\alpha_n$ or α_n+1 we see that $mn+(n+1)(1+\mathrm{ord}_2 a)-\alpha_{n+1}\ge mn+(1+\mathrm{ord}_2 a)n-\alpha_n$. Hence, by (3.11) we get

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r E^{(a)}_{2^m rt+b}\equiv 0 \pmod{2^{mn+(1+\mathrm{ord}_2 a)n-\alpha_n}}. \tag{3.12}$$

For $n'\ge n+1$ we have $n'-\alpha_{n'}\ge n+1-\alpha_{n+1}$ and so

$$mn'+(1+\mathrm{ord}_2 a)n'-\alpha_{n'}\ge(m+\mathrm{ord}_2 a)(n+1)+n+1-\alpha_{n+1}.$$

Thus, using (3.12) we see that for $n'\ge n+1$,

$$\sum_{r=0}^{n'}\binom{n'}{r}(-1)^r E^{(a)}_{2^m rt+b}\equiv 0 \pmod{2^{m+mn+(n+1)(\mathrm{ord}_2 a+1)-\alpha_{n+1}}}. \tag{3.13}$$

When $2 \mid n$ and $2 \nmid b$, by Theorem 3.1(iii) and the fact $\alpha_{n+1} \geq \alpha_n$ we have

$$\mathrm{ord}_2 A_n(a,b) \geq (2+\mathrm{ord}_2 a)(n+1) - \alpha_n - n \geq (n+1)(1+\mathrm{ord}_2 a) + 1 - \alpha_{n+1}.$$

Thus, it follows from (3.11) that

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r E_{2^m r+b}^{(a)} \equiv 0 \pmod{2^{mn+(n+1)(\mathrm{ord}_2 a+1)-\alpha_{n+1}}}. \tag{3.14}$$

By (3.4) we have

$$A_n(a,b) = \frac{1}{2^n} \sum_{r=0}^{[b/2]} \binom{[b/2]}{r} (-1)^r \sum_{s=0}^{r+n} \binom{r+n}{s} (-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)}. \tag{3.15}$$

From Theorem 3.1(iii) we know that

$$\sum_{s=0}^{r+n} \binom{r+n}{s} (-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)} \equiv 0 \pmod{2^{(2+\mathrm{ord}_2 a)(r+n)-\alpha_{r+n}}}.$$

For $r \in \mathbb{N}$ we have $(2+\mathrm{ord}_2 a)(r+n) - \alpha_{r+n} \geq (1+\mathrm{ord}_2 a)(n+1) + n + 1 - \alpha_{n+1}$. Thus, from the above we deduce that

$$A_n(a,b) \equiv e_n(a,b) \pmod{2^{(1+\mathrm{ord}_2 a)(n+1)+1-\alpha_{n+1}}}. \tag{3.16}$$

Now combining (3.11)-(3.14), (3.16) with (3.8) we derive the result.

Theorem 3.3. *Let a be a nonzero integer, $k, m \in \mathbb{N}$, $m \geq 2$ and $b \in \{0, 1, 2, \dots\}$. Then*

$$E_{2^m k+b}^{(a)} - E_b^{(a)} \equiv \begin{cases} 2^m k(a^3((b-1)^2+5) - a + 2^m k a^3 (b-1)) \pmod{2^{m+4+3\mathrm{ord}_2 a}} & \text{if } 2 \mid a, \\ 2^m ka((b+1)^2 + 4 - 2^m k(b+1)) \pmod{2^{m+4}} & \text{if } 2 \nmid a \text{ and } 2 \mid b, \\ 2^m k(a^2-1) \pmod{2^{m+4}} & \text{if } 2 \nmid ab. \end{cases}$$

Proof. For $s \in \mathbb{N}$ let $\alpha_s \in \mathbb{Z}$ be given by $2^{\alpha_s - 1} \leq s < 2^{\alpha_s}$, and let

$$A_s(a,b) = 2^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r E_{2r+b}^{(a)} \quad \text{and} \quad e_s(a,b) = 2^{-s} \sum_{r=0}^{s} \binom{s}{r} (-1)^r E_{2r+b-2[\frac{b}{2}]}^{(a)}.$$

Since $s(r,1) = (r-1)!$ and $S(j,1) = 1$, taking $n = 1$ and $t = k$ in (3.9) we see that

$$\begin{aligned} E_b^{(a)} - E_{2^m k+b}^{(a)} = 2^m k \Big\{ A_1(a,b) + \sum_{r=2}^{2^{m-1}k} A_r(a,b) \Big(\frac{2^{r-1}}{r} \\ - 2 \sum_{j=2}^{r} \frac{(-1)^j s(r,j) j!}{r!} 2^{r-j} \cdot \frac{2^{j-1}}{j!} \cdot 2^{(m-1)j-m} k^{j-1} \Big) \Big\}. \end{aligned} \tag{3.17}$$

For $j \ge 3$ it is easily seen that $\frac{2^{j-1}}{j!} \cdot 2^{(m-1)j-m} \equiv 0 \pmod 4$. By Theorem 3.1(iii) we have $2^{(1+\mathrm{ord}_2 a)r-\alpha_r} \mid A_r(a,b)$. Thus, for $r \ge 5$ we have $\mathrm{ord}_2 A_r(a,b) \ge (1+\mathrm{ord}_2 a)r - \alpha_r \ge 5\mathrm{ord}_2 a + 5 - \alpha_5 = 2 + 5\mathrm{ord}_2 a$. Set $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. From the definition of Stirling numbers we know that $s(n,2) = (n-1)!H_{n-1}$ for $n \ge 2$. Thus, for $r \ge 2$,

$$\frac{(-1)^2 s(r,2)2!}{r!} 2^{r-2} \cdot \frac{2^{2-1}}{2!} \cdot 2^{m-2}k = H_{r-1}\frac{2^{r-1}}{r} \cdot 2^{m-2}k.$$

Hence, from the above we deduce that

$$\begin{aligned}&E^{(a)}_{2^m k+b} - E^{(a)}_b \\ &\equiv -2^m k\Big(A_1(a,b) + \sum_{r=2}^{2^{m-1}k} A_r(a,b)\Big(\frac{2^{r-1}}{r} - H_{r-1}\frac{2^{r-1}}{r}\cdot 2^{m-1}k\Big)\Big) \pmod{2^{m+4+3\mathrm{ord}_2 a}}.\end{aligned}$$

Set

$$f(r) = H_{r-1}\frac{2^{r-1}}{r} = 2^{\alpha_{r-1}-1}H_{r-1}\cdot\frac{2^{r-\alpha_{r-1}}}{r}.$$

Since $2^{\alpha_{r-1}-1}H_{r-1} \in \mathbb{Z}$ we see that $f(r) \equiv 0 \pmod 4$ for $r \ge 5$. It is easily seen that $f(2) = 1,\ f(3) = 2$ and $f(4) = \frac{11}{3}$. Hence, from the above we deduce that

(3.18)

$$\begin{aligned}&E^{(a)}_{2^m k+b} - E^{(a)}_b + 2^m k\Big\{A_1(a,b) + (1-2^{m-1}k)A_2(a,b)\Big\} \\ &\equiv \begin{cases} 0 \pmod{2^{m+4+3\mathrm{ord}_2 a}} & \text{if } m = 2 \text{ and } k = 1, \\ -2^m k\{(\frac{4}{3} - 2^m k)A_3(a,b) + (2 - \frac{11}{3}\cdot 2^{m-1}k)A_4(a,b)\} \pmod{2^{m+4+3\mathrm{ord}_2 a}} \\ \qquad \text{if } m > 2 \text{ or } k > 1. \end{cases}\end{aligned}$$

From (3.16) we see that

(3.19)

$$A_4(a,b) \equiv e_4(a,b) \pmod{2^{3+5\mathrm{ord}_2 a}} \quad \text{and} \quad A_3(a,b) \equiv e_3(a,b) \pmod{2^{2+4\mathrm{ord}_2 a}}.$$

If $2 \nmid b$, by Theorem 3.1(iii) we have

$$A_4(a,b) = \frac{1}{2^4}\sum_{k=0}^{4}\binom{4}{k}(-1)^k E^{(a)}_{2k+b} \equiv 0 \pmod{2^{3+5\mathrm{ord}_2 a}}.$$

From Theorem 2.1 we have

(3.20)

$$\begin{aligned}&E^{(a)}_0 = 1,\ E^{(a)}_1 = 1-a,\ E^{(a)}_2 = 1-2a,\ E^{(a)}_3 = 1-3a+2a^3, \\ &E^{(a)}_4 = 1-4a+8a^3,\ E^{(a)}_5 = 1-5a+20a^3-16a^5, \\ &E^{(a)}_6 = 1-6a+40a^3-96a^5,\ E^{(a)}_7 = 1-7a+70a^3-336a^5+272a^7, \\ &E^{(a)}_8 = 1-8a+112a^3-896a^5+2176a^7.\end{aligned}$$

Hence, if $2 \mid b$, from (3.19) and (3.20) we deduce that

$$\begin{aligned} A_4(a,b) \equiv e_4(a,b) &= \frac{1}{16}\Big(E_0^{(a)} - 4E_2^{(a)} + 6E_4^{(a)} - 4E_6^{(a)} + E_8^{(a)}\Big) \\ &= 8a^5(17a^2-4) \equiv 0 \pmod{2^{3+5\mathrm{ord}_2 a}}. \end{aligned}$$

Therefore, we always have $A_4(a,b) \equiv 0 \pmod{2^{3+5\mathrm{ord}_2 a}}$. From (3.20) we see that

$$e_3(a,b) = \begin{cases} \frac{1}{8}(E_0^{(a)} - 3E_2^{(a)} + 3E_4^{(a)} - E_6^{(a)}) = 2a^3(6a^2-1) & \text{if } 2 \mid b, \\ \frac{1}{8}(E_1^{(a)} - 3E_3^{(a)} + 3E_5^{(a)} - E_7^{(a)}) = 2a^3(-17a^4+18a^2-1) & \text{if } 2 \nmid b. \end{cases} \tag{3.21}$$

Thus, applying (3.19) we get $A_3(a,b) \equiv e_3(a,b) \equiv (1+(-1)^{ab})a^3 \pmod{2^{2+3\mathrm{ord}_2 a}}$. Therefore, $(\frac{4}{3} - 2^m k)A_3(a,b) \equiv (\frac{4}{3} - 2^m k)(1+(-1)^{ab})a^3 \equiv -4a^3(1+(-1)^{ab})(1+2^{m-2}k) \pmod{2^{4+3\mathrm{ord}_2 a}}$. Hence, by the above and (3.18) we obtain

$$\begin{aligned} E_{2^m k+b}^{(a)} - E_b^{(a)} \equiv -2^m k\Big\{&A_1(a,b) + (1-2^{m-1}k)A_2(a,b) \\ &- 4a^3(1+(-1)^{ab})(1+2^{m-2}k)\Big\} \pmod{2^{m+4+3\mathrm{ord}_2 a}}. \end{aligned} \tag{3.22}$$

From Theorem 3.1(ii) we see that

$$\sum_{s=0}^{r}\binom{r}{s}(-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)} \equiv 0 \pmod{2^{7+5\mathrm{ord}_2 a}} \quad \text{for} \quad r \ge 4. \tag{3.23}$$

Thus, by (3.15) we have

$$\begin{aligned} A_2(a,b) &= \frac{1}{4}\sum_{k=0}^{[b/2]}\binom{[b/2]}{k}(-1)^k \sum_{s=0}^{k+2}\binom{k+2}{s}(-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)} \\ &\equiv \frac{1}{4}\Big(\sum_{s=0}^{2}\binom{2}{s}(-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)} - \big[\frac{b}{2}\big]\sum_{s=0}^{3}\binom{3}{s}(-1)^s E_{2s+b-2[\frac{b}{2}]}^{(a)}\Big) \\ &= \frac{1}{4}\Big(4e_2(a,b) - \big[\frac{b}{2}\big]\cdot 8e_3(a,b)\Big) = e_2(a,b) - 2\big[\frac{b}{2}\big]e_3(a,b) \pmod{2^{4+3\mathrm{ord}_2 a}}. \end{aligned}$$

From (3.20) we know that

$$e_2(a,b) = \frac{1}{4}\big(E_{b-2[\frac{b}{2}]}^{(a)} - 2E_{2+b-2[\frac{b}{2}]}^{(a)} + E_{4+b-2[\frac{b}{2}]}^{(a)}\big) = \begin{cases} 2a^3 & \text{if } 2 \mid b, \\ 4a^3(1-a^2) & \text{if } 2 \nmid b. \end{cases} \tag{3.24}$$

This together with (3.21) yields

$$\begin{aligned} &e_2(a,b) - 2\big[\frac{b}{2}\big]e_3(a,b) \\ &= \begin{cases} 2a^3 - b\cdot 2a^3(6a^2-1) = 2a^3(1+b-6a^2b) & \text{if } 2 \mid b, \\ 4a^3(1-a^2) - (b-1)\cdot 2a^3(-17a^4+18a^2-1) & \\ \qquad = 2a^3(17a^4(b-1) + (16-18b)a^2 + b + 1) & \text{if } 2 \nmid b. \end{cases} \end{aligned}$$

Thus,

$$A_2(a,b) \equiv e_2(a,b) - 2\big[\frac{b}{2}\big]e_3(a,b)$$

(3.25)
$$\equiv \begin{cases} 2a^3(1+b) \pmod{2^{4+3\mathrm{ord}_2 a}} & \text{if } 2 \mid a, \\ 2a(1-b) \pmod{2^{4+3\mathrm{ord}_2 a}} & \text{if } 2 \nmid a \text{ and } 2 \mid b, \\ 0 \pmod{2^{4+3\mathrm{ord}_2 a}} & \text{if } 2 \nmid ab. \end{cases}$$

By (3.23) and (3.15) we have

$$\begin{aligned} A_1(a,b) &= \frac{1}{2}\sum_{k=0}^{[b/2]}\binom{[b/2]}{k}(-1)^k\sum_{s=0}^{k+1}\binom{k+1}{s}(-1)^s E^{(a)}_{2s+b-2[\frac{b}{2}]} \\ &\equiv \frac{1}{2}\sum_{k=0}^{2}\binom{[b/2]}{k}(-1)^k\sum_{s=0}^{k+1}\binom{k+1}{s}(-1)^s E^{(a)}_{2s+b-2[\frac{b}{2}]} \\ &= \frac{1}{2}\Big(2e_1(a,b) - \big[\frac{b}{2}\big]\cdot 4e_2(a,b) + \binom{[\frac{b}{2}]}{2}\cdot 8e_3(a,b)\Big) \\ &= e_1(a,b) - 2\big[\frac{b}{2}\big]e_2(a,b) + 2\big[\frac{b}{2}\big]\big(\big[\frac{b}{2}\big]-1\big)e_3(a,b) \pmod{2^{4+3\mathrm{ord}_2 a}}. \end{aligned}$$

From (3.20) we have

(3.26)
$$e_1(a,b) = \frac{1}{2}\big(E^{(a)}_{b-2[\frac{b}{2}]} - E^{(a)}_{2+b-2[\frac{b}{2}]}\big) = \begin{cases} a & \text{if } 2 \mid b, \\ a-a^3 & \text{if } 2 \nmid b \end{cases}$$

Hence, from the above we deduce

(3.27)
$$\begin{aligned} A_1(a,b) &\equiv e_1(a,b) - 2\big[\frac{b}{2}\big]e_2(a,b) + 2\big[\frac{b}{2}\big]\big(\big[\frac{b}{2}\big]-1\big)e_3(a,b) \\ &= \begin{cases} a - b\cdot 2a^3 + \dfrac{b(b-2)}{2}\cdot 2a^3(6a^2-1) = a(1-a^2b^2+6a^4b(b-2)) \\ \qquad \equiv a - a^3b^2 \pmod{2^{4+3\mathrm{ord}_2 a}} \quad \text{if } 2 \mid b, \\ a - a^3 - (b-1)\cdot 4a^3(1-a^2) + (b-1)(b-3)a^3(-17a^4+18a^2-1) \\ \qquad \equiv a - a^3b^2 \pmod{2^{4+3\mathrm{ord}_2 a}} \quad \text{if } 2 \mid a \text{ and } 2 \nmid b, \\ a - a^3 - (b-1)\cdot 4a^3(1-a^2) + (b-1)(b-3)a^3(-17a^4+18a^2-1) \\ \qquad \equiv 1 - a^2 \pmod{2^{4+3\mathrm{ord}_2 a}} \quad \text{if } 2 \nmid a \text{ and } 2 \nmid b. \end{cases} \end{aligned}$$

Now substituting (3.25) and (3.27) into (3.22) we obtain the result.

For $a = 1$, $2 \mid b$ and $m \geq 4$, by Theorem 3.3 we have

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0,6 \pmod 8, \\ E_b - 3\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2,4 \pmod 8. \end{cases}$$

This has been given by the author in [13].

Corollary 3.2. *Let k be a nonnegative integer. Then*

$$E^{(2)}_{4k}\equiv\begin{cases}56k+1\pmod{512} & \text{if } 4\mid k(k-1),\\ 56k-255\pmod{512} & \text{if } 4\nmid k(k-1),\end{cases}$$
$$E^{(2)}_{4k+2}\equiv\begin{cases}-200k-3\pmod{512} & \text{if } 4\mid k(k-1),\\ -200k+253\pmod{512} & \text{if } 4\nmid k(k-1),\end{cases}$$
$$E^{(2)}_{4k+1}\equiv 152k-1\pmod{512},\quad E^{(2)}_{4k+3}\equiv 24k+11\pmod{512}.$$

Proof. Taking $a=2$ and $m=2$ in Theorem 3.3 we deduce the result for $k\ge 1$. Since $E^{(2)}_0=1$, $E^{(2)}_1=-1$, $E^{(2)}_2=-3$ and $E^{(3)}_3=11$, we see that the result is also true for $k=0$.

Theorem 3.4. *Let a be a nonzero integer, $k,m\in\mathbb{N}$ and $b\in\{0,1,2,\ldots\}$. Then*

$$E^{(a)}_{2^mk+b}\equiv kE^{(a)}_{2^m+b}-(k-1)E^{(a)}_b \pmod{2^{2m+1+3\mathrm{ord}_2a}}.$$

Proof. For $s\in\mathbb{N}$ set $e_s(a,b)=2^{-s}\sum_{r=0}^{s}\binom{s}{r}(-1)^rE^{(a)}_{2r+b-2[\frac b2]}$. From (3.24) we know that $2a^3\mid e_2(a,b)$ and so $2^{1+3\mathrm{ord}_2a}\mid e_2(a,b)$. Now taking $n=2$ and $t=1$ in Theorem 3.2 and then applying the above we deduce the result.

4. Congruences for $E^{(2)}_{k(p-1)+b}$ and $E^{(3)}_{k(p-1)+b}\pmod{p^n}$

Let p be an odd prime. In [12] the author showed that $f(k)=(1-(-1)^{\frac{p-1}{2}}p^{k(p-1)+b})\,E_{k(p-1)+b}$ is a p-regular function when b is even. In this section we establish similar results for $E^{(2)}_n$ and $E^{(3)}_n$, and then use them to deduce congruences for $E^{(2)}_{k(p-1)+b}$ and $E^{(3)}_{k(p-1)+b}\pmod{p^n}$.

Lemma 4.1. *Let $m\in\mathbb{N}$, $r\in\{0,1,2,\ldots,m-1\}$ and $b\in\{0,1,2,\ldots\}$. Let p be an odd prime not dividing m. Then*

$$f(k)=m^{k(p-1)+b}\Big(E_{k(p-1)+b}\Big(\frac rm\Big)-(-1)^{\frac{pA_r-r}{m}}E_{k(p-1)+b}\Big(\frac{A_r}{m}\Big)p^{k(p-1)+b}\Big)$$

is a p-regular function, where $A_r\in\{0,1,\ldots,m-1\}$ is given by $pA_r\equiv r\pmod m$.

Proof. For $x\in\mathbb{Z}_p$ let $\langle -x\rangle_p$ be the least nonnegative residue of $-x$ modulo p. From [10,Theorem 3.1] we know that

$$\frac{B_{k(p-1)+b+1}(x)-B_{k(p-1)+b+1}}{k(p-1)+b+1}-p^{k(p-1)+b}\frac{B_{k(p-1)+b+1}(\frac{x+\langle -x\rangle_p}{p})-B_{k(p-1)+b+1}}{k(p-1)+b+1}$$

is a p-regular function. Hence

$$g(k)=\frac{B_{k(p-1)+b+1}(\frac{m+r}{2m})-B_{k(p-1)+b+1}(\frac{r}{2m})}{k(p-1)+b+1}$$
$$-p^{k(p-1)+b}\frac{B_{k(p-1)+b+1}(\frac{\frac{m+r}{2m}+\langle-\frac{m+r}{2m}\rangle_p}{p})-B_{k(p-1)+b+1}(\frac{\frac{r}{2m}+\langle-\frac{r}{2m}\rangle_p}{p})}{k(p-1)+b+1}$$

is a p-regular function.

Let $A_r'\in\{0,1,\ldots,2m-1\}$ be such that $pA_r'\equiv r\pmod{2m}$. Then $p(A_r'\pm m)\equiv m+r\pmod{2m}$ and $A_r=A_r'$ or $A_r'-m$ according as $A_r'<m$ or $A_r'\ge m$. As $\frac{\frac{r}{2m}+\langle-\frac{r}{2m}\rangle_p}{p}=\frac{\frac{r}{2m}+\frac{pA_r'-r}{2m}}{p}=\frac{A_r'}{2m}$ and

$$\frac{\frac{m+r}{2m}+\langle-\frac{m+r}{2m}\rangle_p}{p}=\begin{cases}\dfrac{\frac{m+r}{2m}+\frac{p(A_r'+m)-(m+r)}{2m}}{p}=\dfrac{A_r'+m}{2m} & \text{if } A_r'<m,\\[2ex] \dfrac{\frac{m+r}{2m}+\frac{p(A_r'-m)-(m+r)}{2m}}{p}=\dfrac{A_r'-m}{2m} & \text{if } A_r'\ge m,\end{cases}$$

using (2.1) we see that

$$\frac{B_{k(p-1)+b+1}(\frac{\frac{m+r}{2m}+\langle-\frac{m+r}{2m}\rangle_p}{p})-B_{k(p-1)+b+1}(\frac{\frac{r}{2m}+\langle-\frac{r}{2m}\rangle_p}{p})}{k(p-1)+b+1}$$
$$=\begin{cases}\dfrac{B_{k(p-1)+b+1}(\frac{A_r'+m}{2m})-B_{k(p-1)+b+1}(\frac{A_r'}{2m})}{k(p-1)+b+1}=2^{-(k(p-1)+b+1)}E_{k(p-1)+b}(\dfrac{A_r'}{m})\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{if } A_r'<m,\\[1ex] \dfrac{B_{k(p-1)+b+1}(\frac{A_r'-m}{2m})-B_{k(p-1)+b+1}(\frac{A_r'}{2m})}{k(p-1)+b+1}=-2^{-(k(p-1)+b+1)}E_{k(p-1)+b}(\dfrac{A_r'-m}{m})\\ \qquad\qquad\qquad\qquad\qquad\qquad\text{if } A_r'\ge m.\end{cases}$$
$$=2^{-(k(p-1)+b+1)}\cdot(-1)^{\frac{pA_r-r}{m}}E_{k(p-1)+b}\Big(\frac{A_r}{m}\Big).$$

Also, by (2.1) we have

$$\frac{B_{k(p-1)+b+1}(\frac{m+r}{2m})-B_{k(p-1)+b+1}(\frac{r}{2m})}{k(p-1)+b+1}=2^{-(k(p-1)+b+1)}E_{k(p-1)+b}\Big(\frac{r}{m}\Big).$$

Thus, from the above we see that

$$g(k)=2^{-(k(p-1)+b+1)}\Big(E_{k(p-1)+b}\Big(\frac{r}{m}\Big)-(-1)^{\frac{pA_r-r}{m}}E_{k(p-1)+b}\Big(\frac{A_r}{m}\Big)p^{k(p-1)+b}\Big)$$

is a p-regular function. By Fermat's little theorem we have

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(2m)^{k(p-1)+b}=(2m)^b(1-(2m)^{p-1})^n\equiv 0 \pmod{p^n}.$$

Thus $2^{k(p-1)+b+1}m^{k(p-1)+b}$ is a p-regular function. Hence, by the product theorem for p-regular functions ([10, Theorem 2.3]), $f(k)=2^{k(p-1)+b+1}m^{k(p-1)+b}g(k)$ is a p-regular function as asserted.

From Lemma 4.1 we have the following result.

Lemma 4.2. *Let p be an odd prime, $m\in\{2,3,4,\dots\}$ and $p\equiv\pm1 \pmod m$. Let b be a nonnegative integer and $r\in\{1,2,\dots,m-1\}$. Then*

$$f(k)=\begin{cases}(1-(-1)^{\frac{r(p-1)}{m}}p^{k(p-1)+b})m^{k(p-1)+b}E_{k(p-1)+b}(\frac{r}{m}) & \text{if } p\equiv 1 \pmod m,\\ (1-(-1)^{b+1+\frac{r(p+1)}{m}}p^{k(p-1)+b})m^{k(p-1)+b}E_{k(p-1)+b}(\frac{r}{m}) & \text{if } p\equiv -1 \pmod m\end{cases}$$

is a p-regular function.

Proof. Let $A_r\in\{1,2,\dots,m-1\}$ be such that $pA_r\equiv r \pmod m$. Then clearly $A_r=r$ or $m-r$ according as $p\equiv 1$ or $-1 \pmod m$. Since $E_n(1-x)=(-1)^nE_n(x)$, we have

$$E_{k(p-1)+b}\Big(\frac{m-r}{m}\Big)=(-1)^{k(p-1)+b}E_{k(p-1)+b}\Big(\frac{r}{m}\Big)=(-1)^bE_{k(p-1)+b}\Big(\frac{r}{m}\Big).$$

Now applying the above and Lemma 4.1 we deduce the result.

Theorem 4.1. *Let p be an odd prime and let b be a nonnegative integer. Then*

(i) $f_2(k)=(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{k(p-1)+b})E^{(2)}_{k(p-1)+b}$ *is a p-regular function.*

(ii) $f_3(k)=(1-(-1)^{[\frac{p+1}{6}]}(\frac{p}{3})^{b+1}p^{k(p-1)+b})E^{(3)}_{k(p-1)+b}$ *is a p-regular function.*

Proof. Putting $m=4$ and $r=1$ in Lemma 4.2 and applying Theorem 2.1 we obtain (i). Putting $m=6$ and $r=1$ in Lemma 4.2 and applying Theorem 2.1 we obtain (ii) in the case $p>3$. From Theorem 3.1(i) we see that (ii) is also true for $p=3$. So the theorem is proved.

From Theorem 4.1 and [12, Theorem 4.3 (with $t=1$ and $d=0$)] we deduce the following results.

Theorem 4.2. *Let p be an odd prime and $k,m,n\in\mathbb{N}$. Let b be a nonnegative integer. Then*

$$\Big(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{k\varphi(p^m)+b}\Big)E^{(2)}_{k\varphi(p^m)+b}$$
$$\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\Big(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{r\varphi(p^m)+b}\Big)E^{(2)}_{r\varphi(p^m)+b} \pmod{p^{mn}}$$

and

$$\Big(1-(-1)^{[\frac{p+1}{6}]}\Big(\frac{p}{3}\Big)^{b+1}p^{k\varphi(p^m)+b}\Big)E^{(3)}_{k\varphi(p^m)+b}$$

$$\equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\Big(1-(-1)^{[\frac{p+1}{6}]}\big(\frac{p}{3}\big)^{b+1}p^{r\varphi(p^m)+b}\Big)E^{(3)}_{r\varphi(p^m)+b} \pmod{p^{mn}}.$$

In particular, for $n=1$ we have $E^{(2)}_{k\varphi(p^m)+b}\equiv (1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^b)E^{(2)}_b \pmod{p^m}$ and $E^{(3)}_{k\varphi(p^m)+b}\equiv (1-(-1)^{[\frac{p+1}{6}]}\big(\frac{p}{3}\big)^{b+1}p^b)E^{(3)}_b \pmod{p^m}$.

Lemma 4.3. *(See [10, Theorem 2.1].) Let p be a prime, $n\in\mathbb{N}$ and let f be a p-regular function. Then there are integers $a_0,a_1,\ldots,a_{n-1}$ such that*

$$f(k)\equiv a_{n-1}k^{n-1}+\cdots+a_1k+a_0 \pmod{p^n} \quad \text{for} \quad k=0,1,2,\ldots.$$

Moreover, if $p\ge n$, then $a_0,a_1,\ldots,a_{n-1} \pmod{p^n}$ are uniquely determined and $p^{s-\mathrm{ord}_p s!}\mid a_s$ for $s=0,1,\ldots,n-1$.

From Theorem 4.1 and Lemma 4.3 we deduce the following result.

Theorem 4.3. *Let p be an odd prime, $n\in\mathbb{N}$ and $p\ge n$. Let b be a nonnegative integer. Then there are unique integers $a_0,a_1,\ldots,a_{n-1},c_0,c_1,\ldots,c_{n-1}\in\{0,\pm1,\pm2,\ldots,\pm\frac{p^n-1}{2}\}$ such that for every nonnegative integer k,*

$$\big(1-(-1)^{\frac{p-1}{2}b+[\frac{p-1}{4}]}p^{k(p-1)+b}\big)E^{(2)}_{k(p-1)+b}\equiv a_{n-1}k^{n-1}+\cdots+a_1k+a_0 \pmod{p^n}$$

and

$$\Big(1-(-1)^{[\frac{p+1}{6}]}\Big(\frac{p}{3}\Big)^{b+1}p^{k(p-1)+b}\Big)E^{(3)}_{k(p-1)+b}\equiv c_{n-1}k^{n-1}+\cdots+c_1k+c_0 \pmod{p^n}.$$

Moreover, $p^{s-\mathrm{ord}_p s!}\mid a_s$ and $p^{s-\mathrm{ord}_p s!}\mid c_s$ for $s=0,1,\ldots,n-1$.

Corollary 4.1. *Let $k\in\mathbb{N}$. Then*

(i) $E^{(2)}_{2k}\equiv -9k^2+6k \pmod{27}$, $E^{(2)}_{2k+1}\equiv 9k^2+6k-4 \pmod{27}$;

(ii) $E^{(2)}_{4k}\equiv 1375k^3-375k^2+305k+2 \pmod{3125}$ $(k\ge 2)$;

(iii) $E^{(2)}_{4k+1}\equiv -625k^3-1475k^2-1380k-6 \pmod{3125}$;

(iv) $E^{(2)}_{4k+2}\equiv -375k^3-975k^2-1335k-78 \pmod{3125}$;

(v) $E^{(2)}_{4k+3}\equiv -1500k^3+825k^2-1100k+1386 \pmod{3125}$.

Proof. As $E^{(2)}_0=1$, $E^{(2)}_1=-1$ and $E^{(2)}_2=-3$, taking $p=n=3$ in Theorem 4.3 we see that $\big(1-3^{2k}\big)E^{(2)}_{2k}\equiv -9k^2+6k \pmod{27}$ and $\big(1+3^{2k+1}\big)E^{(2)}_{2k+1}\equiv 9k^2+6k-4 \pmod{27}$. This yields (i). Parts (ii)-(v) can be proved similarly.

Corollary 4.2. *Let $k\in\mathbb{N}$. Then*

(i) $E^{(3)}_{2k}\equiv -6k+1 \pmod{27}$, $E^{(3)}_{2k+1}\equiv -6k-2 \pmod{27}$;

(ii) $E^{(3)}_{4k}\equiv 375k^3+450k^2-620k \pmod{3125}$;

(iii) $E^{(3)}_{4k+1}\equiv 1250k^4-625k^3-175k^2-675k-12 \pmod{3125}$;

(iv) $E^{(3)}_{4k+2}\equiv -625k^4-500k^3+1000k^2-385k+120 \pmod{3125}$;

(v) $E^{(3)}_{4k+3}\equiv -625k^4-625k^3-525k^2-1435k-454 \pmod{3125}$.

5. $\{(-1)^n E_{2n}^{(a)}\}$ is realizable

Let $\{b_n\}(n \geq 1)$ be a given sequence of integers, and let $\{a_n\}$ be defined by $a_1 = b_1$ and $na_n = b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1$ $(n = 2, 3, 4, \ldots)$. If $\{a_n\}$ is also a sequence of integers, following [11] we say that $\{b_n\}$ is a Newton-Euler sequence.

Lemma 5.1. *(See [14, Lemma 5.1].) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. Then the following statements are equivalent:*

(i) *$\{b_n\}$ is a Newton-Euler sequence.*

(ii) $\sum_{d|n} \mu(\frac{n}{d}) b_d \equiv 0 \pmod{n}$ *for every $n \in \mathbb{N}$.*

(iii) *For any prime p and $\alpha, m \in \mathbb{N}$ with $p \nmid m$ we have $b_{mp^{\alpha}} \equiv b_{mp^{\alpha-1}} \pmod{p^{\alpha}}$.*

(iv) *For any $n, t \in \mathbb{N}$ and prime p with $p^t \parallel n$ we have $b_n \equiv b_{\frac{n}{p}} \pmod{p^t}$.*

(v) *There exists a sequence $\{c_n\}$ of integers such that $b_n = \sum_{d|n} dc_d$ for any $n \in \mathbb{N}$.*

Proof. From [1, Theorem 3] or [2] we know that (i), (ii) and (iii) are equivalent. Clearly (iii) is equivalent to (iv). Using Möbius inversion formula we see that (ii) and (v) are equivalent. So the lemma is proved.

Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of nonnegative integers. If there is a set X and a map $T : X \to X$ such that b_n is the number of fixed points of T^n, following [7] and [1] we say that $\{b_n\}$ is realizable.

In [7], Puri and Ward proved that a sequence $\{b_n\}$ of nonnegative integers is realizable if and only if for any $n \in \mathbb{N}$, $\frac{1}{n}\sum_{d|n} \mu(\frac{n}{d}) b_d$ is a nonnegative integer. Thus, using Möbius inversion formula we see that a sequence $\{b_n\}$ is realizable if and only if there exists a sequence $\{c_n\}$ of nonnegative integers such that $b_n = \sum_{d|n} dc_d$ for any $n \in \mathbb{N}$. In [1] J. Arias de Reyna showed that $\{E_{2n}\}$ is a Newton-Euler sequence and $\{|E_{2n}|\}$ is realizable.

Lemma 5.2. *(See [6, p.30]). For $n \in \mathbb{N}$ and $0 \leq x \leq 1$ we have*

$$E_n(x) = 4 \cdot \frac{n!}{\pi^{n+1}} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi x - \frac{n\pi}{2})}{(2m+1)^{n+1}}.$$

Taking $x = \frac{1}{4}$ in Lemma 5.2 and applying Theorem 2.1 we deduce

$$\sum_{m=0}^{\infty} (-1)^{[\frac{m-n}{2}]} \frac{1}{(2m+1)^{n+1}} = \frac{\sqrt{2}E_n^{(2)}}{n!} \left(\frac{\pi}{4}\right)^{n+1}. \tag{5.1}$$

Theorem 5.1. *Let $a, n \in \mathbb{N}$. Then*

$$(-1)^n E_{2n}^{(a)} > \frac{4^{n+1} a^{2n-1} \cdot (2n)!}{\pi^{2n+1}} \left(1 - \frac{1}{(2a+1)^{2n+1}}\right) > 0$$

and

$$(-1)^n E_{2n}^{(a)} < \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\sum_{r=0}^{a-1}\frac{1}{(2r+1)^{2n+1}} < \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\Big(1+\frac{a-1}{3^{2n+1}}\Big).$$

Proof. By Theorem 2.1 and Lemma 5.2 we have

$$\begin{aligned}
&(-1)^n E_{2n}^{(a)}\\
&= (-1)^n(2a)^{2n}E_{2n}\Big(\frac{1}{2a}\Big) = (-4a^2)^n\cdot 4\cdot\frac{(2n)!}{\pi^{2n+1}}\sum_{m=0}^{\infty}\frac{\sin(\frac{(2m+1)\pi}{2a}-n\pi)}{(2m+1)^{2n+1}}\\
&= \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\sum_{m=0}^{\infty}\frac{\sin\frac{(2m+1)\pi}{2a}}{(2m+1)^{2n+1}} = \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\sum_{r=0}^{2a-1}\frac{\sin\frac{(2r+1)\pi}{2a}}{(4ak+2r+1)^{2n+1}}\\
&= \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\sum_{r=0}^{a-1}\sin\frac{(2r+1)\pi}{2a}\Big(\frac{1}{(4ak+2r+1)^{2n+1}}-\frac{1}{(4ak+2a+2r+1)^{2n+1}}\Big).
\end{aligned}$$

For $r\in\{0,1,\dots,a-1\}$ we have $\sin\frac{(2r+1)\pi}{2a}>0$ and so

$$\sin\frac{(2r+1)\pi}{2a}\Big(\frac{1}{(4ak+2r+1)^{2n+1}}-\frac{1}{(4ak+2a+2r+1)^{2n+1}}\Big)>0.$$

Thus,

$$(-1)^n E_{2n}^{(a)} > \frac{4^{n+1}a^{2n}\cdot(2n)!}{\pi^{2n+1}}\Big(1-\frac{1}{(2a+1)^{2n+1}}\Big)\sin\frac{\pi}{2a}.$$

It is well known that $\sin x\ge\frac{2}{\pi}x$ for $0\le x\le\frac{\pi}{2}$. Thus $\sin\frac{\pi}{2a}\ge\frac{1}{a}$. So the first inequality is true. Since

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{r=0}^{a-1}\sin\frac{(2r+1)\pi}{2a}\Big(\frac{1}{(4ak+2r+1)^{2n+1}}-\frac{1}{(4ak+2a+2r+1)^{2n+1}}\Big)\\
&<\sum_{r=0}^{a-1}\sum_{k=0}^{\infty}\Big(\frac{1}{(4ak+2r+1)^{2n+1}}-\frac{1}{(4ak+4a+2r+1)^{2n+1}}\Big)\\
&=\sum_{r=0}^{a-1}\frac{1}{(2r+1)^{2n+1}}<1+(a-1)\frac{1}{3^{2n+1}},
\end{aligned}$$

combining the above we obtain the remaining inequality.

Theorem 5.2. *Let $n\in\mathbb{N}$ with $2\nmid n$. Then $(-1)^{\frac{n+1}{2}}E_n^{(2)}>0$ and $(-1)^{\frac{n+1}{2}}E_n^{(3)}>0$.*

Proof. For $k\ge 0$ we see that

$$\frac{1}{(8k+1)^{n+1}}-\frac{1}{(8k+3)^{n+1}}-\frac{1}{(8k+5)^{n+1}}+\frac{1}{(8k+7)^{n+1}}$$

$$
\begin{aligned}
&= \frac{(8k+1)^{n+1}+(8k+7)^{n+1}}{(8k+1)^{n+1}(8k+7)^{n+1}} - \frac{(8k+3)^{n+1}+(8k+5)^{n+1}}{(8k+3)^{n+1}(8k+5)^{n+1}} \\
&> \frac{(8k+1)^{n+1}+(8k+7)^{n+1}-(8k+3)^{n+1}-(8k+5)^{n+1}}{(8k+3)^{n+1}(8k+5)^{n+1}} \\
&= \frac{\sum_{r=0}^{n}\binom{n+1}{r}(8k+3)^r 4^{n+1-r} - \sum_{r=0}^{n}\binom{n+1}{r}(8k+1)^r 4^{n+1-r}}{(8k+3)^{n+1}(8k+5)^{n+1}} > 0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(-1)^{\frac{n+1}{2}}\sum_{m=0}^{\infty}(-1)^{[\frac{m-n}{2}]}\frac{1}{(2m+1)^{n+1}} \\
&= \sum_{k=0}^{\infty}\Big(\frac{1}{(8k+1)^{n+1}} - \frac{1}{(8k+3)^{n+1}} - \frac{1}{(8k+5)^{n+1}} + \frac{1}{(8k+7)^{n+1}}\Big) > 0.
\end{aligned}
$$

Now applying (5.1) we deduce $(-1)^{\frac{n+1}{2}}E_n^{(2)} > 0$.

Similarly, for $k \ge 0$ we have

$$
\frac{1}{(12k+1)^{n+1}} - \frac{1}{(12k+5)^{n+1}} - \frac{1}{(12k+7)^{n+1}} + \frac{1}{(12k+11)^{n+1}} > 0.
$$

Thus, using Lemma 5.2 and Theorem 2.1 we obtain

$$
\begin{aligned}
&(-1)^{\frac{n+1}{2}}\frac{E_n^{(3)}\cdot \pi^{n+1}}{4\cdot 6^n\cdot n!} \\
&= (-1)^{\frac{n+1}{2}}\frac{E_n\big(\frac16\big)\pi^{n+1}}{4\cdot n!} = (-1)^{\frac{n+1}{2}}\sum_{m=0}^{\infty}\frac{\sin(2m+1-3n)\frac{\pi}{6}}{(2m+1)^{n+1}} \\
&= \frac{\sqrt{3}}{2}\sum_{k=0}^{\infty}\Big(\frac{1}{(12k+1)^{n+1}} - \frac{1}{(12k+5)^{n+1}} - \frac{1}{(12k+7)^{n+1}} + \frac{1}{(12k+11)^{n+1}}\Big) > 0.
\end{aligned}
$$

Hence, $(-1)^{\frac{n+1}{2}}E_n^{(3)} > 0$. The proof is now complete.

Theorem 5.3. *Let a be a positive integer. For any prime divisor p of $n \in \mathbb{N}$ we have $E_{2n}^{(a)} \equiv E_{2n/p}^{(a)} \pmod{p^{\mathrm{ord}_p n}}$. Hence $\{E_{2n}^{(a)}\}$ is a Newton-Euler sequence.*

Proof. Suppose $2 \mid n$ and $n = 2^m n_0$ with $2 \nmid n_0$. From Theorem 3.3 we see that $E_{2^m k}^{(a)} \equiv E_0^{(a)} = 1 \pmod{2^m}$ for $m \ge 2$ and $k \in \mathbb{N}$. It is well known that $2 \nmid E_{2k}$. Thus, using Theorem 2.1 we see that $E_{2k}^{(a)} \equiv (1-a)^{2k}E_0 + a^{2k}E_{2k} \equiv 1 \pmod 2$. Hence,

$$
E_{2n}^{(a)} = E_{2^m\cdot 2n_0}^{(a)} \equiv 1 \equiv E_{2^m n_0}^{(a)} = E_n^{(a)} \pmod{2^m}.
$$

Now suppose that p is an odd prime divisor of n and $n = p^m n_1$ with $p \nmid n_1$. If $p \mid a$, by Theorem 2.1 and the fact $\frac{p^{2r-1}}{r} \in \mathbb{Z}_p$ for $r \ge 1$ we have

$$
E_{2n}^{(a)} = (1-a)^{2n} + \sum_{r=1}^{n} n\binom{2n-1}{2r-1}(1-a)^{2n-2r}\frac{a^{2r}}{r}E_{2r} \equiv (1-a)^{2n} \pmod{p^m}
$$

and

$$E^{(a)}_{\frac{2n}{p}} = (1-a)^{\frac{2n}{p}} + \sum_{r=1}^{n/p} \frac{n}{p}\binom{2\frac{n}{p}-1}{2r-1}(1-a)^{2\frac{n}{p}-2r}\frac{a^{2r}}{r}E_{2r} \equiv (1-a)^{\frac{2n}{p}} \pmod{p^m}.$$

Since $p \mid a$, we have

$$(1-a)^{2n} = (1-a)^{2n_1\varphi(p^m)+2n_1p^{m-1}} \equiv (1-a)^{2n_1p^{m-1}} = (1-a)^{\frac{2n}{p}} \pmod{p^m}.$$

Thus $E^{(a)}_{2n} \equiv E^{(a)}_{2n/p} \pmod{p^m}$.

Let us consider the case $p \nmid a$. Suppose that $A_1 \in \{0,1,\ldots,2a-1\}$ is given by $2a \mid (pA_1-1)$. From Lemma 4.1 we know that for a given nonnegative integer b,

$$f(k) = (2a)^{k(p-1)+b}E_{k(p-1)+b}\Big(\frac{1}{2a}\Big) - (-1)^{\frac{pA_1-1}{2a}}(2a)^{k(p-1)+b}E_{k(p-1)+b}\Big(\frac{A_1}{2a}\Big)p^{k(p-1)+b}$$

is a p-regular function. By [10, Corollary 2.1] we have $f(kp^{m-1}) \equiv f(0) \pmod{p^m}$. Thus, using Theorem 2.1 we obtain

$$E^{(a)}_{kp^{m-1}(p-1)+b} \equiv E^{(a)}_b \pmod{p^m} \quad \text{for} \quad b \geq m. \tag{5.2}$$

As $2n_1p^{m-1} \geq m$, using (5.2) we see that

$$E^{(a)}_{2n} = E^{(a)}_{2n_1p^m} = E^{(a)}_{2n_1p^{m-1}(p-1)+2n_1p^{m-1}} \equiv E^{(a)}_{2n_1p^{m-1}} = E^{(a)}_{2n/p} \pmod{p^m}.$$

Now putting all the above together with Lemma 5.1 we obtain the result.

Theorem 5.4. *Let $a \in \mathbb{N}$. Then $\{(-1)^nE^{(a)}_{2n}\}$ is realizable.*

Proof. Suppose that p is a prime divisor of n and $t = \mathrm{ord}_p n$. From Theorem 5.3 we know that $E^{(a)}_{2n} \equiv E^{(a)}_{2n/p} \pmod{p^t}$. It is easily seen that $(-1)^n \equiv (-1)^{n/p} \pmod{p^t}$. Thus, $(-1)^nE^{(a)}_{2n} \equiv (-1)^{n/p}E^{(a)}_{2n/p} \pmod{p^t}$. Hence, using Lemma 5.1 we know that $\{(-1)^nE^{(a)}_{2n}\}$ is a Newton-Euler sequence and so $\frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})(-1)^dE^{(a)}_{2d} \in \mathbb{Z}$. By Theorem 5.1, $(-1)^nE^{(a)}_{2n} > 0$. Now it remains to show that $\sum_{d|n}\mu(\frac{n}{d})(-1)^dE^{(a)}_{2d} \geq 0$. From (3.20) we have $E^{(a)}_2 = 1-2a$ and $E^{(a)}_4 = 1-4a+8a^3$. Thus the inequality is true for $n = 1,2$.

From now on we assume $n \geq 3$. Observe that $1+\frac{a-1}{3^{2m+1}} < a$ and $1-\frac{1}{(2a+1)^{2m+1}} \geq 1-\frac{1}{3^{2m+1}} \geq 1-\frac{1}{27} = \frac{26}{27}$ for $m \in \mathbb{N}$. Using Theorem 5.1 we see that for $m \in \mathbb{N}$,

$$\frac{26}{27}\cdot\frac{4^{m+1}a^{2m-1}\cdot(2m)!}{\pi^{2m+1}} < (-1)^mE^{(a)}_{2m} < \frac{4^{m+1}a^{2m+1}\cdot(2m)!}{\pi^{2m+1}}. \tag{5.3}$$

Hence

$$\sum_{d|n}\mu(n/d)(-1)^dE^{(a)}_{2d}$$

$$
\begin{aligned}
&= (-1)^n E_{2n}^{(a)} + \sum_{d|n, d\le \frac{n}{2}} \mu(n/d)(-1)^d E_{2d}^{(a)} \\
&\ge (-1)^n E_{2n}^{(a)} - \sum_{d=1}^{[n/2]} (-1)^d E_{2d}^{(a)} > \frac{26}{27} \cdot \frac{4^{n+1} a^{2n-1} \cdot (2n)!}{\pi^{2n+1}} - \sum_{d=1}^{[n/2]} \frac{4^{d+1} a^{2d+1} \cdot (2d)!}{\pi^{2d+1}} \\
&\ge \frac{26}{27} \cdot \frac{16a}{\pi^3} \cdot (2n)! \Big(\frac{4a^2}{\pi^2}\Big)^{n-1} - \sum_{d=1}^{[n/2]} \frac{4a}{\pi} \Big(\frac{4a^2}{\pi^2}\Big)^d \cdot n! \\
&= \frac{26}{27} \cdot \frac{16a}{\pi^3} \cdot n! \Big\{ (n+1)(n+2)\cdots(2n) \Big(\frac{4a^2}{\pi^2}\Big)^{n-1} - \frac{27}{26} \cdot \frac{\pi^2}{4} \cdot \frac{(\frac{4a^2}{\pi^2})^{[\frac{n}{2}]+1} - \frac{4a^2}{\pi^2}}{\frac{4a^2}{\pi^2} - 1} \Big\}.
\end{aligned}
$$

For $a \ge 2$ we have $(\frac{4a^2}{\pi^2})^{n-1} > (\frac{4a^2}{\pi^2})^{[\frac{n}{2}]+1} - \frac{4a^2}{\pi^2}$ and

$$
(n+1)(n+2)\cdots(2n) \ge 4 \cdot 5 \cdot 6 > \frac{27}{26} \cdot \frac{\pi^2/4}{16/\pi^2 - 1} \ge \frac{27}{26} \cdot \frac{\pi^2}{4} \cdot \frac{1}{4a^2/\pi^2 - 1}.
$$

Thus, from the above we deduce $\sum_{d|n} \mu\big(\frac{n}{d}\big)(-1)^d E_{2d}^{(a)} > 0$. For $a = 1$ we see that

$$
\begin{aligned}
&(n+1)(n+2)\cdots(2n) \Big(\frac{4}{\pi^2}\Big)^{n-1} - \frac{27}{26} \cdot \frac{\pi^2}{4} \cdot \frac{\frac{4}{\pi^2} - (\frac{4}{\pi^2})^{[\frac{n}{2}]+1}}{1 - \frac{4}{\pi^2}} \\
&> 2n \Big(\frac{4(n+1)}{\pi^2}\Big)^{n-1} - \frac{27}{26} \cdot \frac{1}{1 - 4/\pi^2} > 2n - \frac{27}{26} \cdot \frac{1}{1 - 4/\pi^2} > 0
\end{aligned}
$$

and so $\sum_{d|n} \mu\big(\frac{n}{d}\big)(-1)^d E_{2d}^{(a)} > 0$ by the above.

Now summarizing the above we prove the theorem.

Acknowledgements. The author thanks the referee for his helpful comments and valuable suggestion on improving Theorem 3.3.

References

[1] J. Arias de Reyna, *Dynamical zeta functions and Kummer congruences*, Acta Arith. **119** (2005), 39-52.

[2] B.S. Du, S.S. Huang and M.C. Li, *Generalized Fermat, double Fermat and Newton sequences*, J. Number Theory **98** (2003), 172-183.

[3] R. Ernvall *A congruence on Euler numbers*, Amer. Math. Monthly **89** (1982), 431.

[4] H.W. Gould, *Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, Rev. Ed., Morgantown Printing and Binding Co., West-Virginia, 1972.

[5] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory* (*2nd edition*), Springer, New York, 1990, pp. 233,248.

[6] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (*3rd edition*), Springer, New York, 1966, pp. 25-32.

[7] Y. Puri and T. Ward, *Arithmetic and growth of periodic orbits*, J. Integer Seq. **4**(2001), Art. 01.2.1, 18pp.

[8] M.A. Stern, *Zur Theorie der Eulerschen Zahlen*, J. Reine Angew. Math. **79** (1875), 67-98.

[9] Z.H. Sun, *Congruences for Bernoulli numbers and Bernoulli polynomials*, Discrete Math. **163** (1997), 153-163.

[10] Z.H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105**(2000), 193-223.

[11] Z.H. Sun, *On the properties of Newton-Euler pairs*, J. Number Theory **114**(2005), 88-123.

[12] Z.H. Sun, *Congruences involving Bernoulli polynomials*, Discrete Math. **308**(2008), 71-112.

[13] Z.H. Sun, *Euler numbers modulo* 2^n, Bull. Aust. Math. Soc. **82** (2010), 221-231.

[14] Z.H. Sun, *Identities and congruences for a new sequence*, Int. J. Number Theory, to appear.

[15] J. Urbanowicz and K.S. Williams, *Congruences for L-Functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[16] S.S. Wagstaff Jr., *Prime divisors of the Bernoulli and Euler numbers,* in: M.A. Bennett et al. (Eds.), Number Theory for the Millennium, vol. III (Urbana, IL, 2000), A K Peters, 2002, pp. 357-374.