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SUPERCONGRUENCES INVOLVING EULER POLYNOMIALS

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ABSTRACT. Let $p>3$ be a prime, and let a be a rational p-adic integer. Let $\{E_n(x)\}$ denote the Euler polynomials given by $\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}$. In this paper we show that

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\equiv(-1)^{\langle a\rangle_p}+(a-\langle a\rangle_p)(p+a-\langle a\rangle_p)E_{p-3}(-a)\pmod{p^3},$$
$$\sum_{k=0}^{p-1}\binom{a}{k}(-2)^k\equiv(-1)^{\langle a\rangle_p}-(a-\langle a\rangle_p)E_{p-2}(-a)\pmod{p^2}\quad\text{for}\quad a\not\equiv0\pmod p,$$

where $\langle a\rangle_p\in\{0,1,\ldots,p-1\}$ satisfying $a\equiv\langle a\rangle_p\pmod p$. Taking $a=-\frac13,-\frac14,-\frac16$ in the first congruence we solve some conjectures of Z.W. Sun. We also establish a congruence for $\sum_{k=0}^{p-1}k\binom{a}{k}\binom{-1-a}{k}$ modulo p^3.

1. Introduction

Let $p>3$ be a prime. In 2003, based on his work concerning hypergeometric functions and Calabi-Yau manifolds, Rodriguez-Villegas [RV] posed 22 conjectures on supercongruences. The following congruences are 8 conjectures of Rodriguez-Villegas:

$$(1.1)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2},$$

$$(1.2)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{p-1}\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

$$(1.3)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv0\pmod{p^2}\quad\text{for}\quad p\equiv3\pmod 4,$$

$$(1.4)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv0\pmod{p^2}\quad\text{for}\quad p\equiv5\pmod 6,$$

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$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2} \quad\text{for}\quad p\equiv 5,7 \pmod 8, \tag{1.5}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv 0 \pmod{p^2} \quad\text{for}\quad p\equiv 3 \pmod 4, \tag{1.6}$$

where $(\frac{a}{p})$ is the Legendre symbol. Here (1.1) and (1.2) were later confirmed by Mortenson [M1-M2], (1.3) was first conjectured by Beukers [Be] in 1987 and proved by van Hamme [vH]. (1.4)-(1.6) were finally proved by Z. W. Sun [Su2]. (1.1)-(1.6) are concerned with Legendre polynomials and elliptic curves over finite fields. See [S5, S8-S10]. For the progress on other conjectures of Rodriguez-Villegas see [Mc].

The Bernoulli numbers $\{B_n\}$ and Bernoulli polynomials $\{B_n(x)\}$ are defined by

$$B_0=1,\ \sum_{k=0}^{n-1}\binom{n}{k}B_k=0\ (n\ge 2)\quad\text{and}\quad B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\ (n\ge 0).$$

The Euler numbers $\{E_n\}$ and Euler polynomials $\{E_n(x)\}$ are defined by

$$E_0=1,\ E_n=-\sum_{k=1}^{[n/2]}\binom{n}{2k}E_{n-2k}\ (n\ge 1)\ \text{and}\ E_n(x)=\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(2x-1)^{n-k}E_k,$$

where $[a]$ is the greatest integer not exceeding a. It is well known that $B_{2n+1}=0$ and $E_{2n-1}=0$ for any positive integer n. $\{B_n\}$ and $\{E_n\}$ are important sequences and they have many interesting properties and applications. See [EMOT], [MOS], [Sl, A000111] and [S1-S4]. By [Sl], $|E_{2n}|$ is the number of permutations $a_1a_2\cdots a_{2n}$ on $1,2,\ldots,2n$ such that $a_1>a_2<a_3>\cdots<a_{2n-1}>a_{2n}$. Euler showed that (see [MOS])

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^{2n+1}}=\frac{(-1)^nE_{2n}}{2\cdot(2n)!}\Big(\frac{\pi}{2}\Big)^{2n+1}$$

and

$$\sum_{r=0}^{m-1}(-1)^r r^n=\frac{E_n(0)-(-1)^mE_n(m)}{2}\quad\text{for any positive integers } m \text{ and } n,$$

and Ernvall [E] proved that

$$E_{(p-1)/2}\equiv 2h(-4p) \pmod p \quad\text{for any prime}\quad p\equiv 1 \pmod 4,$$

where $h(d)$ is the class number of the form class group consisting of classes of primitive, integral binary quadratic forms of discriminant d.

Let $p>3$ be a prime. In [Su1], using a complicated method the author's brother Z.W. Sun proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\Big(\frac{-1}{p}\Big)-p^2E_{p-3} \pmod{p^3} \tag{1.7}$$

and conjectured that (see [Su1, Conjecture 5.12])

$$(1.8)\qquad \sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\Big(\frac{-1}{p}\Big)-\frac{25}{9}p^2E_{p-3}\pmod{p^3},$$

$$(1.9)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\Big(\frac{-2}{p}\Big)-\frac{3}{16}p^2E_{p-3}\Big(\frac{1}{4}\Big)\pmod{p^3},$$

$$(1.10)\qquad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}\equiv\Big(\frac{-3}{p}\Big)-\frac{p^2}{3}B_{p-2}\Big(\frac{1}{3}\Big)\pmod{p^3}.$$

As pointed out in [S11], we have

$$(1.11)\qquad \begin{aligned}&\binom{-\frac{1}{2}}{k}^2=\frac{\binom{2k}{k}^2}{16^k},\quad \binom{-\frac{1}{3}}{k}\binom{-\frac{2}{3}}{k}=\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},\\ &\binom{-\frac{1}{4}}{k}\binom{-\frac{3}{4}}{k}=\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\quad \binom{-\frac{1}{6}}{k}\binom{-\frac{5}{6}}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{432^k}.\end{aligned}$$

Let $\mathbb{Z}$ be the set of integers. For a prime p let $\mathbb{Z}_p$ denote the set of rational $p-adic$ integers. For a $p-adic$ integer a let $\langle a\rangle_p\in\{0,1,\dots,p-1\}$ be given by $a\equiv\langle a\rangle_p \pmod p$. Let p be a prime greater than 3 and $a\in\mathbb{Z}_p$. In [S11] the author showed that

$$(1.12)\qquad \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\equiv(-1)^{\langle a\rangle_p}\pmod{p^2}.$$

Taking $a=-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ in (1.12) and then applying (1.11) we get (1.1)-(1.2) immediately. In [S11], the author showed that

$$(1.13)\qquad \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\binom{2k}{k}\frac{1}{4^k}\equiv 0\pmod{p^2}\quad\text{for}\quad \langle a\rangle_p\equiv 1\pmod 2.$$

Taking $a=-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ in (1.13) and then applying (1.11) we deduce (1.3)-(1.6).

Let $p>3$ be a prime and $a\in\mathbb{Z}_p$. In this paper we improve (1.12) by showing that

$$(1.14)\qquad \sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k}\equiv(-1)^{\langle a\rangle_p}+(a-\langle a\rangle_p)(p+a-\langle a\rangle_p)E_{p-3}(-a)\pmod{p^3}.$$

Taking $a=-\frac{1}{3},-\frac{1}{4},-\frac{1}{6}$ in (1.14) we deduce Z.W. Sun's conjectures (1.8)-(1.10). We also determine $\sum_{k=0}^{p-1}k\binom{a}{k}\binom{-1-a}{k}$ modulo p^3 and prove that for $a\not\equiv 0\pmod p$,

$$(1.15)\qquad \sum_{k=0}^{p-1}\binom{a}{k}(-2)^k\equiv(-1)^{\langle a\rangle_p}-(a-\langle a\rangle_p)E_{p-2}(-a)\pmod{p^2}.$$

Throughout this paper $H_m=1+\frac{1}{2}+\dots+\frac{1}{m}$ for $m=1,2,3,\dots$.

2. Congruences for $\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \pmod{p^3}$

Lemma 2.1. *Let $p>3$ be a prime and $t\in\mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\equiv 1 \pmod{p^3}.$$

Proof. For $k\in\{1,2,\ldots,p-1\}$ we see that

$$\begin{aligned}\binom{pt}{k}\binom{-1-pt}{k}&=\frac{pt(pt-1)\cdots(pt-k+1)(-1-pt)(-2-pt)\cdots(-k-pt)}{k!^2}\\&=\frac{(-1)^k pt(pt+k)}{k!^2}(p^2t^2-1^2)\cdots(p^2t^2-(k-1)^2)\\&\equiv-\frac{pt(pt+k)}{k^2}=-\frac{p^2t^2}{k^2}-\frac{pt}{k}\pmod{p^3}.\end{aligned}$$

From [L] or [S2] we know that

$$\sum_{k=1}^{p-1}\frac{1}{k^2}\equiv 0\pmod p\quad\text{and}\quad\sum_{k=1}^{p-1}\frac1k\equiv 0\pmod{p^2}. \tag{2.1}$$

Thus,

$$\sum_{k=0}^{p-1}\binom{pt}{k}\binom{-1-pt}{k}\equiv 1-p^2t^2\sum_{k=1}^{p-1}\frac{1}{k^2}-pt\sum_{k=1}^{p-1}\frac1k\equiv 1\pmod{p^3}.$$

This proves the lemma.

Lemma 2.2. *Let p be an odd prime, $a\in\mathbb{Z}_p$, $a\not\equiv 0 \pmod p$ and $k\in\{1,2,\ldots,p-2\}$. Then*

$$\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r^k}\equiv-\frac{(2^{p-k}-1)B_{p-k}}{p-k}+\frac12(-1)^{\langle a\rangle_p+k}E_{p-1-k}(-a)\pmod p.$$

Proof. For positive integers m and n it is well known ([MOS]) that $\sum_{r=0}^{m-1}(-1)^r r^n=\frac{E_n(0)-(-1)^mE_n(m)}{2}$. Thus,

$$\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r^k}\equiv\sum_{r=0}^{\langle a\rangle_p}(-1)^r r^{p-1-k}=\frac{E_{p-1-k}(0)-(-1)^{\langle a\rangle_p+1}E_{p-1-k}(\langle a\rangle_p+1)}{2}\pmod p.$$

From [MOS] and [S6, (2.2)-(2.3)] we know that

$$E_n(0)=\frac{2(1-2^{n+1})B_{n+1}}{n+1}\quad\text{and}\quad E_n(1-x)=(-1)^nE_n(x). \tag{2.2}$$

Hence,

$$\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r^k}\equiv-\frac{(2^{p-k}-1)B_{p-k}}{p-k}+\frac12(-1)^{\langle a\rangle_p+k}E_{p-1-k}(-\langle a\rangle_p)\pmod p.$$

Set $a = \langle a\rangle_p + pt$. It is well known ([MOS]) that $E_n(x+y) = \sum_{s=0}^{n}\binom{n}{s}x^sE_{n-s}(y)$. Thus,

$$\begin{aligned}E_{p-1-k}(-\langle a\rangle_p) = E_{p-1-k}(pt-a) &= \sum_{s=0}^{p-1-k}\binom{p-1-k}{s}(pt)^sE_{p-1-k-s}(-a)\\ &\equiv E_{p-1-k}(-a) \pmod p.\end{aligned}$$

We are done.

Lemma 2.3 ([S11, Lemma 4.2]). *Let p be an odd prime, $m \in \{1,2,\ldots,p-1\}$ and $t \in \mathbb{Z}_p$. Then*

$$\binom{m+pt-1}{p-1} \equiv \frac{pt}{m} - \frac{p^2t^2}{m^2} + \frac{p^2t}{m}H_m \pmod{p^3}.$$

Theorem 2.1. *Let $p > 3$ be a prime and $a \in \mathbb{Z}_p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \equiv (-1)^{\langle a\rangle_p} + (a-\langle a\rangle_p)(p+a-\langle a\rangle_p)E_{p-3}(-a) \pmod{p^3}.$$

Moreover, for $a \not\equiv 0 \pmod p$ we have

$$\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \equiv (-1)^{\langle a\rangle_p} + (a-\langle a\rangle_p)(p+a-\langle a\rangle_p)\Big(\frac{2}{a^2} - E_{p-3}(a)\Big) \pmod{p^3}.$$

Proof. For given positive integer n set $S_n(x) = \sum_{k=0}^{n}\binom{x}{k}\binom{-1-x}{k}$. Since $\binom{x}{k}\binom{-1-x}{k} + \binom{x+1}{k}\binom{-2-x}{k} = 2(\binom{x}{k}\binom{-2-x}{k} - \binom{x}{k-1}\binom{-2-x}{k-1})$ for $k = 1,2,\ldots$, we see that

$$\begin{aligned}S_n(x) + S_n(x+1) &= 2 + 2\sum_{k=1}^{n}\Big(\binom{x}{k}\binom{-2-x}{k} - \binom{x}{k-1}\binom{-2-x}{k-1}\Big)\\ &= 2\binom{x}{n}\binom{-2-x}{n} = 2(-1)^n\binom{x}{n}\binom{x+1+n}{n}.\end{aligned}$$

When $a = pt \equiv 0 \pmod p$, from the proof of Lemma 2.2 we have $E_{p-3}(-pt) \equiv E_{p-3}(0) = 2(1-2^{p-2})B_{p-2}/(p-2) = 0 \pmod p$. Thus, the result follows from Lemma 2.1. Now suppose that $a \not\equiv 0 \pmod p$ and $a = \langle a\rangle_p + pt$. Then $t \in \mathbb{Z}_p$ and $a - k = \langle a\rangle_p - k + pt$. From the above identity we see that

$$\begin{aligned}&S_n(a) - (-1)^{\langle a\rangle_p}S_n(pt)\\ &= \sum_{k=0}^{\langle a\rangle_p-1}(-1)^k(S_n(a-k-1) + S_n(a-k)) = 2\sum_{k=0}^{\langle a\rangle_p-1}(-1)^{n+k}\binom{a-k-1}{n}\binom{a-k+n}{n}.\end{aligned}$$

Hence applying Lemma 2.3 we deduce that

$$\begin{aligned}&S_{p-1}(a) - (-1)^{\langle a\rangle_p}S_{p-1}(pt)\\ &= 2\sum_{k=0}^{\langle a\rangle_p-1}(-1)^{p-1+k}\binom{\langle a\rangle_p - k + pt - 1}{p-1}\binom{\langle a\rangle_p - k + p(t+1) - 1}{p-1}\end{aligned}$$

$$
\begin{aligned}
&\equiv 2\sum_{k=0}^{\langle a\rangle_p-1}(-1)^k\Big(\frac{pt}{\langle a\rangle_p-k}-\frac{p^2t^2}{(\langle a\rangle_p-k)^2}+\frac{p^2t}{\langle a\rangle_p-k}H_{\langle a\rangle_p-k}\Big)\\
&\quad\times\Big(\frac{p(t+1)}{\langle a\rangle_p-k}-\frac{p^2(t+1)^2}{(\langle a\rangle_p-k)^2}+\frac{p^2(t+1)}{\langle a\rangle_p-k}H_{\langle a\rangle_p-k}\Big)\\
&\equiv 2\sum_{k=0}^{\langle a\rangle_p-1}(-1)^k\Big(\frac{p^2t(t+1)}{(\langle a\rangle_p-k)^2}-\frac{p^3t(t+1)(2t+1)}{(\langle a\rangle_p-k)^3}+2\frac{p^3t(t+1)H_{\langle a\rangle_p-k}}{(\langle a\rangle_p-k)^2}\Big)\\
&\equiv 2\sum_{r=1}^{\langle a\rangle_p}(-1)^{\langle a\rangle_p-r}\Big(\frac{p^2t(t+1)}{r^2}-\frac{p^3t(t+1)(2t+1)}{r^3}+2\frac{p^3t(t+1)H_r}{r^2}\Big)\pmod{p^4}.
\end{aligned}
$$

As $B_{2m+1}=0$ for $m\ge 1$, we see that $B_{p-2}=0$. Thus, by Lemma 2.2 we have $\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r^2}\equiv\frac12(-1)^{\langle a\rangle_p}E_{p-3}(-a)\pmod p$. Now, from the above and Lemma 2.1 we deduce that

$$
\begin{aligned}
S_{p-1}(a)&\equiv(-1)^{\langle a\rangle_p}S_{p-1}(pt)+(-1)^{\langle a\rangle_p}2p^2t(t+1)\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r^2}\\
&\equiv(-1)^{\langle a\rangle_p}+p^2t(t+1)E_{p-3}(-a)\pmod{p^3}.
\end{aligned}
$$

It is well known that ([MOS]) $E_n(1-x)=(-1)^nE_n(x)$ and $E_n(x)+E_n(x+1)=2x^n$. Thus, $E_{p-3}(-a)=E_{p-3}(1+a)=2a^{p-3}-E_{p-3}(a)\equiv\frac{2}{a^2}-E_{p-3}(a)\pmod p$. Recall that $t=(a-\langle a\rangle_p)/p$. By the above, the theorem is proved.

Taking $a=-\frac12$ in Theorem 2.1 and then applying (1.11) and the fact $E_n=2^nE_n(\frac12)$ we obtain (1.7).

For $m=3,4,6$ it is clear that

$$
(2.3)\qquad -\frac1m-\langle-\frac1m\rangle_p=\begin{cases}-\frac1m-\frac{p-1}{m}=-\frac pm & \text{if } p\equiv 1\pmod m,\\ -\frac1m-\frac{(m-1)p-1}{m}=-\frac{(m-1)p}{m} & \text{if } p\equiv -1\pmod m\end{cases}
$$

and so

$$
(2.4)\qquad \Big(-\frac1m-\langle-\frac1m\rangle_p\Big)\Big(p-\frac1m-\langle-\frac1m\rangle_p\Big)=-\frac pm\cdot\frac{(m-1)p}{m}=-\frac{m-1}{m^2}p^2.
$$

Corollary 2.1. *Let $p>3$ be a prime. Then*

$$
\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}\equiv\Big(\frac{-1}{p}\Big)-\frac{25}{9}p^2E_{p-3}\pmod{p^3}.
$$

Proof. Taking $a=-\frac16$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$
\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{p-1} \binom{-\frac{1}{6}}{k}\binom{-\frac{5}{6}}{k} \equiv (-1)^{\langle -\frac{1}{6}\rangle_p} + \Big(-\frac{1}{6} - \big\langle -\frac{1}{6}\big\rangle_p\Big)\Big(p - \frac{1}{6} - \big\langle -\frac{1}{6}\big\rangle_p\Big) E_{p-3}\Big(\frac{1}{6}\Big) \\
&\equiv \Big(\frac{-1}{p}\Big) - \frac{5}{36} E_{p-3}\Big(\frac{1}{6}\Big) \pmod{p^3}.
\end{aligned}
$$

By [S6, Theorem 2.1 and Lemma 2.1], we have $6^{2n}E_{2n}(\frac{1}{6}) = \frac{3^{2n}+1}{2}E_{2n}$. Thus, $E_{p-3}(\frac{1}{6}) = \frac{1}{6^{p-3}} \cdot \frac{3^{p-3}+1}{2}E_{p-3} \equiv 20E_{p-3} \pmod p$. Hence the result follows.

In [S7] the author introduced the sequence $\{U_n\}$ given by

$$U_0 = 1 \quad \text{and} \quad U_n = -2\sum_{k=1}^{[n/2]} \binom{n}{2k} U_{n-2k} \ (n \ge 1).$$

Clearly $U_{2n-1} = 0$. For any prime $p > 3$, in [S7] the author proved that $\sum_{k=1}^{[2p/3]} \frac{(-1)^{k-1}}{k} \equiv 3p(\frac{p}{3})U_{p-3} \pmod{p^2}$.

Corollary 2.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \Big(\frac{-3}{p}\Big) - 2p^2 U_{p-3} \pmod{p^3}.$$

Proof. Taking $a = -\frac{1}{3}$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$
\begin{aligned}
&\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \\
&= \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}\binom{-\frac{2}{3}}{k} \equiv (-1)^{\langle -\frac{1}{3}\rangle_p} + \Big(-\frac{1}{3} - \big\langle -\frac{1}{3}\big\rangle_p\Big)\Big(p - \frac{1}{3} - \big\langle -\frac{1}{3}\big\rangle_p\Big) E_{p-3}\Big(\frac{1}{3}\Big) \\
&= \Big(\frac{-3}{p}\Big) - \frac{2}{9} E_{p-3}\Big(\frac{1}{3}\Big) \pmod{p^3}.
\end{aligned}
$$

By [S7, Theorem 2.1], $U_{2n} = 3^{2n}E_{2n}(\frac{1}{3})$. Thus, $U_{p-3} = 3^{p-3}E_{p-3}(\frac{1}{3}) \equiv \frac{1}{9}E_{p-3}(\frac{1}{3}) \pmod p$. Now putting all the above together we obtain the result.

Remark 2.1. Let $p > 3$ be a prime. By [S7, p.217], $B_{p-2}(\frac{1}{3}) \equiv 6U_{p-3} \pmod p$. Thus, from Corollary 2.2 we deduce (1.10). In [MT], Mattarei and Tauraso proved that $\sum_{k=0}^{p-1}\binom{2k}{k} \equiv (\frac{-3}{p}) - \frac{p^2}{3}B_{p-2}(\frac{1}{3}) \pmod{p^3}$. This together with Corollary 2.2 yields

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k} \equiv \Big(\frac{-3}{p}\Big) - 2p^2 U_{p-3} \pmod{p^3}.$$

In [S3] the author introduced the sequence $\{S_n\}$ given by $S_0 = 1$ and $S_n = 1 - \sum_{k=0}^{n-1}\binom{n}{k}2^{2n-2k-1}S_k$ ($n \ge$ 1), and showed that $S_n = 4^n E_n(\frac{1}{4})$.

Corollary 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\equiv\Big(\frac{-2}{p}\Big)-3p^2S_{p-3}\pmod{p^3}.$$

Proof. Taking $a=-\frac14$ in Theorem 2.1 and then applying (1.11) and (2.4) we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\\&=\sum_{k=0}^{p-1}\binom{-\frac14}{k}\binom{-\frac34}{k}\equiv(-1)^{\langle-\frac14\rangle_p}+\Big(-\frac14-\Big\langle-\frac14\Big\rangle_p\Big)\Big(p-\frac14-\Big\langle-\frac14\Big\rangle_p\Big)E_{p-3}\Big(\frac14\Big)\\&=\Big(\frac{-2}{p}\Big)-\frac{3}{16}E_{p-3}\Big(\frac14\Big)\pmod{p^3}.\end{aligned}$$

Since $S_{p-3}=4^{p-3}E_{p-3}(\frac14)\equiv\frac1{16}E_{p-3}(\frac14)\pmod p$, we obtain the result.

Lemma 2.4. *For any nonnegative integer n we have*

$$\sum_{k=0}^{n}(k-a(a+1))\binom{a}{k}\binom{-1-a}{k}=-a(a+1)\binom{a-1}{n}\binom{-2-a}{n}.$$

Proof. Observe that

$$\begin{aligned}&-a(a+1)\Big\{\binom{a-1}{n+1}\binom{-2-a}{n+1}-\binom{a-1}{n}\binom{-2-a}{n}\Big\}\\&=\binom{a}{n+1}\binom{-1-a}{n+1}((a-n-1)(-2-a-n)-(n+1)^2)\\&=(n+1-a(a+1))\binom{a}{n+1}\binom{-1-a}{n+1}.\end{aligned}$$

The result can be easily proved by induction on n.

Theorem 2.2. *Let $p>3$ be a prime and $a\in\mathbb{Z}_p$ with $a\not\equiv0,-1\pmod p$. Then*

$$\sum_{k=0}^{p-1}k\binom{a}{k}\binom{-1-a}{k}\equiv(-1)^{\langle a\rangle_p}a(a+1)+p^2t(t+1)\big(a(a+1)E_{p-3}(-a)-1\big)\pmod{p^3},$$

where $t=(a-\langle a\rangle_p)/p$.

Proof. By Lemma 2.3, we have $\binom{a-1}{p-1}=\binom{\langle a\rangle_p+pt-1}{p-1}\equiv\frac{pt}{\langle a\rangle_p}\pmod{p^2}$ and

$$\binom{-2-a}{p-1}=\binom{p-1-\langle a\rangle_p-p(t+1)-1}{p-1}\equiv\frac{p(-t-1)}{p-1-\langle a\rangle_p}\equiv\frac{p(t+1)}{\langle a\rangle_p+1}\pmod{p^2}.$$

Thus,

$$\binom{a-1}{p-1}\binom{-2-a}{p-1}\equiv\frac{t(t+1)}{\langle a\rangle_p(\langle a\rangle_p+1)}p^2\equiv\frac{t(t+1)}{a(a+1)}p^2\pmod{p^3}.$$

Hence, using Lemma 2.4 we see that

$$\begin{aligned}
&\sum_{k=0}^{p-1} k\binom{a}{k}\binom{-1-a}{k} - a(a+1)\sum_{k=0}^{p-1}\binom{a}{k}\binom{-1-a}{k} \\
&= -a(a+1)\binom{a-1}{p-1}\binom{-2-a}{p-1} \equiv -p^2t(t+1) \pmod{p^3}.
\end{aligned} \tag{2.5}$$

This together with Theorem 2.1 yields the result.

Theorem 2.3. *Let $p>3$ be a prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{432^k} &\equiv -\frac{5}{36}\Big(\frac{-1}{p}\Big) + \frac{5}{324}p^2(9+25E_{p-3}) \pmod{p^3},\\
\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k} &\equiv -\frac{2}{9}\Big(\frac{-3}{p}\Big) + \frac{2}{9}p^2(1+2U_{p-3}) \pmod{p^3},\\
\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{64^k} &\equiv -\frac{3}{16}\Big(\frac{-2}{p}\Big) + \frac{3}{16}p^2(1+3S_{p-3}) \pmod{p^3}.
\end{aligned}$$

Proof. Taking $a=-\frac{1}{6},-\frac{1}{3},-\frac{1}{4}$ in (2.5) and then applying (1.11) and Corollaries 2.1-2.3 we deduce the result.

Remark 2.2. For any prime $p>3$, in [Su3, Corollary 1.2 (with $x=1$)] Z.W. Sun obtained congruences for $\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}$, $\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{64^k}$ and $\sum_{k=0}^{p-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{432^k}$ modulo p^2.

3. A congruence for $\sum_{k=0}^{p-1}\binom{a}{k}(-2)^k \pmod{p^2}$

For given positive integer n and variables a and x define

$$S_n(a,x)=\sum_{k=0}^{n}\binom{a}{k}x^k.$$

As $\binom{a}{k}=\binom{a-1}{k}+\binom{a-1}{k-1}$ for $k\ge 1$, we see that

$$\begin{aligned}
S_n(a,x) &= 1+\sum_{k=1}^{n}\binom{a-1}{k}x^k+\sum_{k=1}^{n}\binom{a-1}{k-1}x^k\\
&= S_n(a-1,x)+x\Big(S_n(a-1,x)-\binom{a-1}{n}x^n\Big).
\end{aligned}$$

Thus,

$$S_n(a,x)-(1+x)S_n(a-1,x)=-\binom{a-1}{n}x^{n+1}. \tag{3.1}$$

Therefore,

$$\begin{aligned}&S_n(a,x)-(1+x)^{\langle a\rangle_p}S_n(a-\langle a\rangle_p,x)\\&=\sum_{k=1}^{\langle a\rangle_p}(1+x)^{k-1}(S_n(a-k+1,x)-(1+x)S_n(a-k,x))=-\sum_{k=1}^{\langle a\rangle_p}(1+x)^{k-1}\binom{a-k}{n}x^{n+1}.\end{aligned}$$

Note that $\binom{a-k}{n}=(-1)^n\binom{k-a+n-1}{n}$. We then obtain

$$S_n(a,x)-(1+x)^{\langle a\rangle_p}S_n(a-\langle a\rangle_p,x)=(-x)^{n+1}\sum_{k=1}^{\langle a\rangle_p}(1+x)^{k-1}\binom{n-a+k-1}{n}. \tag{3.2}$$

Let p be an odd prime, $a\in\mathbb{Z}_p$ and $a=\langle a\rangle_p+pt$. Then $t\in\mathbb{Z}_p$. For $1\le k\le\langle a\rangle_p\le n\le p-1$ we see that

$$\begin{aligned}&\binom{n-a+k-1}{n}\\&=\frac{(n-\langle a\rangle_p+k-1-pt)\cdots(1-pt)(-pt)(-1-pt)\cdots(-(\langle a\rangle_p-k)-pt)}{n!}\\&\equiv\frac{(n-\langle a\rangle_p+k-1)!(-pt)(-1)^{\langle a\rangle_p-k}(\langle a\rangle_p-k)!}{n!}=-pt\cdot\frac{(-1)^{\langle a\rangle_p-k}}{n\binom{n-1}{\langle a\rangle_p-k}}\pmod{p^2}.\end{aligned}$$

Thus,

$$\begin{aligned}S_n(a,x)-(1+x)^{\langle a\rangle_p}S_n(pt,x)&\equiv-pt\frac{(-x)^{n+1}}{n}\sum_{k=1}^{\langle a\rangle_p}(1+x)^{k-1}\frac{(-1)^{\langle a\rangle_p-k}}{\binom{n-1}{\langle a\rangle_p-k}}\\&=-pt\frac{(-x)^{n+1}}{n}\sum_{r=0}^{\langle a\rangle_p-1}(1+x)^{\langle a\rangle_p-1-r}\frac{(-1)^r}{\binom{n-1}{r}}\pmod{p^2}.\end{aligned}$$

Since

$$S_n(pt,x)=1+\sum_{k=1}^{n}\frac{pt}{k}\binom{pt-1}{k-1}x^k\equiv1-pt\sum_{k=1}^{n}\frac{(-x)^k}{k}\pmod{p^2},$$

for $a,x\in\mathbb{Z}_p$, $1\le\langle a\rangle_p\le n\le p-1$ and $x\not\equiv-1\pmod p$ we have

$$\begin{aligned}S_n(a,x)\equiv{}&(1+x)^{\langle a\rangle_p}-(a-\langle a\rangle_p)(1+x)^{\langle a\rangle_p}\Big(\sum_{k=1}^{n}\frac{(-x)^k}{k}\\&-\frac{(-x)^{n+1}}{n}\sum_{k=0}^{\langle a\rangle_p-1}\frac{1}{\binom{n-1}{k}(-1-x)^{k+1}}\Big)\pmod{p^2}.\end{aligned}\tag{3.3}$$

Suppose that p is an odd prime, $a\in\mathbb{Z}_p$ and $a=\langle a\rangle_p+pt\not\equiv0\pmod p$. Taking $n=p-1$ in (3.1) and then applying Lemma 2.3 we see that

$$S_{p-1}(a,x)-(x+1)S_{p-1}(a-1,x)$$

$$= -\binom{\langle a\rangle_p + pt - 1}{p-1} x^p \equiv \Big(-\frac{pt}{\langle a\rangle_p} + \frac{p^2t^2}{\langle a\rangle_p^2} - \frac{p^2 t}{\langle a\rangle_p} H_{\langle a\rangle_p}\Big) x^p \pmod{p^3}.$$

For $1 \le k \le \langle a\rangle_p$ we have $\langle a-k+1\rangle_p = \langle a\rangle_p - k + 1$ and so $a-k+1 = \langle a-k+1\rangle_p + pt$. Thus,

$$\begin{aligned}
&S_{p-1}(a,x) - (x+1)^{\langle a\rangle_p} S_{p-1}(a - \langle a\rangle_p, x)\\
&= \sum_{k=1}^{\langle a\rangle_p} (x+1)^{k-1}(S_{p-1}(a-k+1,x) - (x+1)S_{p-1}(a-k,x))\\
&\equiv \sum_{k=1}^{\langle a\rangle_p} (x+1)^{k-1} x^p \Big(-\frac{pt}{\langle a\rangle_p - k + 1} + \frac{p^2t^2}{(\langle a\rangle_p - k+1)^2} - \frac{p^2 t}{\langle a\rangle_p - k + 1} H_{\langle a\rangle_p - k + 1}\Big)\\
&= x^p \sum_{r=1}^{\langle a\rangle_p} (x+1)^{\langle a\rangle_p - r}\Big(-\frac{pt}{r} + \frac{p^2t^2}{r^2} - \frac{p^2t}{r} H_r\Big)\\
&= ptx^p(x+1)^{\langle a\rangle_p}\Big(-\sum_{r=1}^{\langle a\rangle_p} \frac{1}{r(x+1)^r} + pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2(x+1)^r} - p\sum_{r=1}^{\langle a\rangle_p}\frac{H_r}{r(x+1)^r}\Big) \pmod{p^3}.
\end{aligned}$$

Define $H_0 = 0$. For $k \in \{1,2,\dots,p-1\}$ we see that

$$\binom{p}{k} = \frac{p}{k}\cdot\frac{(p-1)\cdots(p-(k-1))}{(k-1)!} \equiv \frac{p}{k}(-1)^{k-1}(1 - pH_{k-1}) \pmod{p^3}$$

and so $\frac{(-1)^{k-1}}{k} \equiv \frac{1}{p}\binom{p}{k} + p\frac{(-1)^{k-1}}{k}H_{k-1} \pmod{p^2}$. Hence

$$\begin{aligned}
\sum_{k=1}^{p-1}\frac{(-x)^k}{k} &\equiv -\sum_{k=1}^{p-1} x^k\Big(\frac{1}{p}\binom{p}{k} + p\frac{(-1)^{k-1}}{k}H_{k-1}\Big)\\
&= -\frac{1}{p}((1+x)^p - 1 - x^p) + p\sum_{k=1}^{p-1}\frac{(-x)^k}{k}H_{k-1} \pmod{p^2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&S_{p-1}(pt,x)\\
&= 1 + \sum_{k=1}^{p-1}\frac{pt}{k}\cdot\frac{(pt-1)\cdots(pt-(k-1))}{(k-1)!}x^k \equiv 1 + \sum_{k=1}^{p-1}\frac{pt}{k}(-1)^{k-1}(1 - ptH_{k-1})x^k\\
&= 1 + pt(t-1)\sum_{k=1}^{p-1}\frac{(-x)^k}{k} + t^2\sum_{k=1}^{p-1}(-1)^{k-1}\frac{p}{k}(1 - pH_{k-1})x^k\\
&\equiv 1 + pt(t-1)\sum_{k=1}^{p-1}\frac{(-x)^k}{k} + t^2\sum_{k=1}^{p-1}\binom{p}{k}x^k\\
&\equiv 1 + t(t-1)\Big(-((1+x)^p - 1 - x^p) + p^2\sum_{k=1}^{p-1}\frac{(-x)^k}{k}H_{k-1}\Big) + t^2((1+x)^p - 1 - x^p)
\end{aligned}$$

$$= 1 + t((1+x)^p - 1 - x^p) + p^2 t(t-1)\sum_{k=1}^{p-1}\frac{(-x)^k}{k}H_{k-1} \pmod{p^3}.$$

Now, from the above we deduce that

$$\begin{aligned}(3.4)\qquad S_{p-1}(a,x) &\equiv (x+1)^{\langle a\rangle_p}\Big(1 + t((1+x)^p - 1 - x^p) + p^2 t(t-1)\sum_{k=1}^{p-1}\frac{(-x)^k}{k}H_{k-1}\Big)\\ &+ ptx^p(x+1)^{\langle a\rangle_p}\Big(-\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r(x+1)^r} + pt\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r^2(x+1)^r} - p\sum_{r=1}^{\langle a\rangle_p}\frac{H_r}{r(x+1)^r}\Big) \pmod{p^3}.\end{aligned}$$

Lemma 3.1. *Let p be an odd prime, $a, x \in \mathbb{Z}_p$, $a(x+1) \not\equiv 0 \pmod p$ and $t = (a - \langle a\rangle_p)/p$. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\binom{a}{k}x^k &\equiv (x+1)^{\langle a\rangle_p}\Big(1 + t((1+x)^p - 1 - x^p) - ptx\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r(x+1)^r}\Big)\\ &\equiv (x+1)^{\langle a\rangle_p}\Big(1 + t((1+x)^p - 1 - x^p) + tx\sum_{r=1}^{\langle a\rangle_p}\binom{p}{r}\Big(-\frac{1}{x+1}\Big)^r\Big) \pmod{p^2}.\end{aligned}$$

Proof. For $r \in \{1, 2, \dots, p-1\}$ we have $\binom{p}{r} = \frac{p}{r}\binom{p-1}{r-1} \equiv \frac{(-1)^{r-1}}{r}p \pmod{p^2}$. Thus,

$$-p\sum_{r=1}^{\langle a\rangle_p}\frac{1}{r(x+1)^r} = \sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^{r-1}}{r}p\Big(-\frac{1}{x+1}\Big)^r \equiv \sum_{r=1}^{\langle a\rangle_p}\binom{p}{r}\Big(-\frac{1}{x+1}\Big)^r \pmod{p^2}.$$

Now the result follows from (3.4).

Theorem 3.1. *Let p be an odd prime, $a \in \mathbb{Z}_p$ and $a \not\equiv 0 \pmod p$. Then*

$$\sum_{k=0}^{p-1}\binom{a}{k}(-2)^k \equiv (-1)^{\langle a\rangle_p} - (a - \langle a\rangle_p)E_{p-2}(-a) \pmod{p^2}.$$

Proof. Set $q_p(2) = (2^{p-1}-1)/p$ and $t = (a - \langle a\rangle_p)/p$. Taking $x = -2$ in Lemma 3.1 we see that

$$\sum_{k=0}^{p-1}\binom{a}{k}(-2)^k \equiv (-1)^{\langle a\rangle_p}(1 + t((-1)^p - 1 - (-2)^p)) - pt(-2)^p(-1)^{\langle a\rangle_p}\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r} \pmod{p^2}.$$

It is well known that $pB_{p-1} \equiv p-1 \pmod p$. Thus, from Lemma 2.2 we deduce that

$$\begin{aligned}\sum_{r=1}^{\langle a\rangle_p}\frac{(-1)^r}{r} &\equiv -\frac{q_p(2)pB_{p-1}}{p-1} + \frac{1}{2}(-1)^{\langle a\rangle_p+1}E_{p-2}(-a)\\ &\equiv -q_p(2) - \frac{1}{2}(-1)^{\langle a\rangle_p}E_{p-2}(-a) \pmod p.\end{aligned}$$

Now combining all the above we deduce that

$$\sum_{k=0}^{p-1}\binom{a}{k}(-2)^k \equiv (-1)^{\langle a\rangle_p}(1+2ptq_p(2))+2pt(-1)^{\langle a\rangle_p}\Big(-q_p(2)-\frac{1}{2}(-1)^{\langle a\rangle_p}E_{p-2}(-a)\Big)$$
$$= (-1)^{\langle a\rangle_p}-ptE_{p-2}(-a) \pmod{p^2}.$$

This proves the theorem.

Theorem 3.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-1/3}{k}(-2)^k \equiv \Big(\frac{-3}{p}\Big)+\frac{3-(\frac{-3}{p})}{3}(2^{p-1}-1) \pmod{p^2}.$$

Proof. Taking $a=-\frac{1}{3}$ in Theorem 3.1 and then applying (2.3) we see that

$$\sum_{k=0}^{p-1}\binom{-1/3}{k}(-2)^k$$
$$\equiv (-1)^{\langle -\frac{1}{3}\rangle_p}-\Big(-\frac{1}{3}-\big\langle -\frac{1}{3}\big\rangle_p\Big)E_{p-2}\Big(\frac{1}{3}\Big)=\Big(\frac{-3}{p}\Big)-\frac{(\frac{-3}{p})-3}{6}pE_{p-2}\Big(\frac{1}{3}\Big) \pmod{p^2}.$$

From [MOS] we know that $B_{2n}\big(\frac{1}{3}\big)=\frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n}$. Now applying [S6, Lemma 2.2] and the well known fact $pB_{p-1}\equiv p-1 \pmod p$ we deduce that

$$E_{p-2}\Big(\frac{1}{3}\Big)$$
$$=\frac{2}{p-1}((-2)^{p-1}-1)B_{p-1}\Big(\frac{1}{3}\Big)=\frac{2}{p-1}(2^{p-1}-1)\cdot\frac{3-3^{p-1}}{2\cdot 3^{p-1}}B_{p-1}\equiv 2\frac{2^{p-1}-1}{p} \pmod p.$$

Thus the result follows.

Remark 3.1 In [Su1], Z.W. Sun proved that for any odd prime p,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}=\sum_{k=0}^{p-1}\binom{-1/2}{k}(-2)^k\equiv\Big(\frac{-1}{p}\Big)-p^2E_{p-3} \pmod{p^3}.$$

This can be deduced from (3.4).

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