

Acta Arith. 159(2013), 1-25.

Congruences for $q^{[p/8]} \pmod{p}$

ZHI-HONG SUN

School of Mathematical Sciences, Huaiyin Normal University,
Huai'an, Jiangsu 223001, PR China

E-mail: zhihongsun@yahoo.com

Homepage: <http://www.hytc.edu.cn/xsjl/szh>

ABSTRACT. Let \mathbb{Z} be the set of integers, and let (m, n) be the greatest common divisor of integers m and n . Let $p \equiv 1 \pmod{4}$ be a prime, $q \in \mathbb{Z}$, $2 \nmid q$ and $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$. Suppose that $(c, x + d) = 1$ or $(d, x + c)$ is a power of 2. In the paper, by using the quartic reciprocity law we determine $q^{[p/8]} \pmod{p}$ in terms of c, d, x and y , where $[\cdot]$ is the greatest integer function. Hence we partially solve some conjectures posed by the author in two previous papers.

MSC: Primary 11A15, Secondary 11A07, 11E25

Keywords: Reciprocity law; octic residue; congruence; quartic Jacobi symbol

1. Introduction.

Let \mathbb{Z} be the set of integers, $i = \sqrt{-1}$ and $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. For any positive odd number m and $a \in \mathbb{Z}$ let $\left(\frac{a}{m}\right)$ be the (quadratic) Jacobi symbol. (We also assume $\left(\frac{a}{1}\right) = 1$.) For our convenience we also define $\left(\frac{a}{-m}\right) = \left(\frac{a}{m}\right)$. Then for any two odd numbers m and n with $m > 0$ or $n > 0$ we have the following general quadratic reciprocity law: $\left(\frac{m}{n}\right) = (-1)^{\frac{m-1}{2} \cdot \frac{n-1}{2}} \left(\frac{n}{m}\right)$.

For $a, b, c, d \in \mathbb{Z}$ with $2 \nmid c$ and $2 \mid d$, one can define the quartic Jacobi symbol $\left(\frac{a+bi}{c+di}\right)_4$ as in [S4]. From [IR] we know that $\overline{\left(\frac{a+bi}{c+di}\right)_4} = \left(\frac{a-bi}{c-di}\right)_4 = \left(\frac{a+bi}{c+di}\right)_4^{-1}$, where \bar{x} means the complex conjugate of x . In Section 2 we list main properties of the quartic Jacobi symbol. See also [IR], [BEW] and [S2].

For a prime $p = 24k + 1 = c^2 + d^2 = x^2 + 3y^2$ with $k, c, d, x, y \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$, in [HW] and [H], by using cyclotomic numbers and Jacobi sums Hudson and Williams proved that

$$3^{\frac{p-1}{8}} \equiv \begin{cases} \pm 1 \pmod{p} & \text{if } c \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}, \\ \pm \frac{d}{c} \pmod{p} & \text{if } d \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}. \end{cases}$$

Let p be a prime of the form $4k + 1$, $q \in \mathbb{Z}$, $2 \nmid q$ and $p \nmid q$. Suppose that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $c \equiv 1 \pmod{4}$. In [S4] and [S5] the author posed many conjectures on $q^{[p/8]} \pmod{p}$ in terms of c, d, x and y , where $[\cdot]$ is the

The author is supported by the National Natural Sciences Foundation of China (No. 10971078).

greatest integer function. For $m, n \in \mathbb{Z}$ let (m, n) be the greatest common divisor of m and n . For $m \in \mathbb{Z}$ with $m = 2^\alpha m_0 (2 \nmid m_0)$ we say that $2^\alpha \parallel m$. In the paper, by developing the calculation technique of quartic Jacobi symbols we partially solve many conjectures from [S4] and [S5], and establish new reciprocity laws for quartic and octic residues on condition that $(c, x + d) = 1$ or $(d, x + c) = 2^\alpha$. For the history of classical reciprocity laws, see [Lem]. Suppose $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x + d) = 1$ or $(d_0, x + c) = 1$. We then have the following typical results.

(1.1) If $p \equiv q \equiv 1 \pmod{8}$, q is a prime and $q = a^2 + b^2$ with $a, b \in \mathbb{Z}$, then

$$q^{\frac{p-1}{8}} \equiv (-1)^{\frac{d}{4} + \frac{xy}{4}} \left(\frac{d}{c}\right)^m \pmod{p} \iff \left(\frac{ac + bd}{ac - bd}\right)^{\frac{q-1}{8}} \equiv \left(\frac{b}{a}\right)^m \pmod{q}.$$

(1.2) If $q \equiv 7 \pmod{8}$ is a prime, then

$$q^{[p/8]} \equiv \begin{cases} (-1)^{\frac{y}{4}} \left(\frac{d}{c}\right)^m \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{x-1}{2}} \left(\frac{d}{c}\right)^m \frac{y}{x} \pmod{p} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

$$\iff \left(\frac{c - di}{c + di}\right)^{\frac{q+1}{8}} \equiv i^m \pmod{q}.$$

(1.3) If $p \equiv 1 \pmod{8}$, $q = a^2 + b^2$, $a, b \in \mathbb{Z}$, $2 \mid a$ and $(a, b) = 1$, then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{d}{4} + \frac{x}{4}} \left(\frac{c}{d}\right)^m \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \mid x, \\ (-1)^{\frac{d}{4} + \frac{y}{4}} \left(\frac{c}{d}\right)^m \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \nmid x, \\ (-1)^{\frac{b-1}{2} + \frac{d}{4} + \frac{x+2}{4}} \left(\frac{c}{d}\right)^{m-1} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \mid x, \\ (-1)^{\frac{b-1}{2} + \frac{d}{4} + \frac{y}{4} + \frac{x-1}{2}} \left(\frac{c}{d}\right)^{m-1} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \nmid x \end{cases}$$

$$\iff \left(\frac{(ac + bd)/x}{b + ai}\right)_4 = i^m.$$

2. Basic lemmas.

Lemma 2.1 ([S4, Proposition 2.1]). *Let $a, b \in \mathbb{Z}$ with $2 \nmid a$ and $2 \mid b$. Then*

$$\left(\frac{i}{a + bi}\right)_4 = i^{\frac{a^2 + b^2 - 1}{4}} = (-1)^{\frac{a^2 - 1}{8}} i^{(1 - (-1)^{\frac{b}{2}})/2}$$

and

$$\left(\frac{1 + i}{a + bi}\right)_4 = \begin{cases} i^{((-1)^{\frac{a-1}{2}}(a-b)-1)/4} & \text{if } 4 \mid b, \\ i^{(-1)^{\frac{a-1}{2}}(b-a)-1} & \text{if } 2 \parallel b. \end{cases}$$

Lemma 2.2 ([S4, Proposition 2.2]). *Let $a, b \in \mathbb{Z}$ with $2 \nmid a$ and $2 \mid b$. Then*

$$\left(\frac{-1}{a + bi}\right)_4 = (-1)^{\frac{b}{2}} \quad \text{and} \quad \left(\frac{2}{a + bi}\right)_4 = i^{(-1)^{\frac{a-1}{2}} \frac{b}{2}}.$$

Lemma 2.3 ([S4, Proposition 2.3]). *Let $a, b, c, d \in \mathbb{Z}$ with $2 \nmid ac$, $2 \mid b$ and $2 \mid d$. If $a+bi$ and $c+di$ are relatively prime elements of $\mathbb{Z}[i]$, we have the following general law of quartic reciprocity:*

$$\left(\frac{a+bi}{c+di}\right)_4 = (-1)^{\frac{b}{2} \cdot \frac{c-1}{2} + \frac{d}{2} \cdot \frac{a+b-1}{2}} \left(\frac{c+di}{a+bi}\right)_4.$$

In particular, if $4 \mid b$, then

$$\left(\frac{a+bi}{c+di}\right)_4 = (-1)^{\frac{a-1}{2} \cdot \frac{d}{2}} \left(\frac{c+di}{a+bi}\right)_4.$$

Lemma 2.4 ([E], [S1, Lemma 2.1]). *Let $a, b, m \in \mathbb{Z}$ with $2 \nmid m$ and $(m, a^2+b^2) = 1$. Then*

$$\left(\frac{a+bi}{m}\right)_4^2 = \left(\frac{a^2+b^2}{m}\right).$$

Lemma 2.5 ([S3, Lemma 4.3]). *Let $a, b \in \mathbb{Z}$ with $2 \nmid a$ and $2 \mid b$. For any integer x with $(x, a^2+b^2) = 1$ we have*

$$\left(\frac{x^2}{a+bi}\right)_4 = \left(\frac{x}{a^2+b^2}\right).$$

Lemma 2.6. *Let $a, b \in \mathbb{Z}$ with $2 \mid b$ and $(a, b) = 1$. Then*

$$\left(\frac{b}{a+bi}\right)_4 = \begin{cases} 1 & \text{if } 4 \mid b, \\ (-1)^{\frac{a-1}{2}} i & \text{if } 2 \parallel b. \end{cases}$$

Proof. By Lemmas 2.1 and 2.3,

$$\begin{aligned} \left(\frac{b}{a+bi}\right)_4 &= \left(\frac{i}{a+bi}\right)_4 \left(\frac{-bi}{a+bi}\right)_4 = \left(\frac{i}{a+bi}\right)_4 \left(\frac{a}{a+bi}\right)_4 \\ &= \left(\frac{i}{a+bi}\right)_4 \cdot (-1)^{\frac{a-1}{2} \cdot \frac{b}{2}} \left(\frac{a+bi}{a}\right)_4 = (-1)^{\frac{a-1}{2} \cdot \frac{b}{2}} \left(\frac{i}{a+bi}\right)_4 \left(\frac{i}{a}\right)_4 \\ &= (-1)^{\frac{a-1}{2} \cdot \frac{b}{2}} \cdot (-1)^{\frac{a^2-1}{8}} i^{\frac{1-(-1)^{b/2}}{2}} \cdot (-1)^{\frac{a^2-1}{8}} \\ &= \begin{cases} 1 & \text{if } 4 \mid b, \\ (-1)^{\frac{a-1}{2}} i & \text{if } 2 \parallel b. \end{cases} \end{aligned}$$

Thus the lemma is proved.

For a given odd prime p let \mathbb{Z}_p denote the set of those rational numbers whose denominator is not divisible by p . Following [S1,S2] we define

$$Q_r(p) = \left\{ k \mid k \in \mathbb{Z}_p, \left(\frac{k+i}{p}\right)_4 = i^r \right\} \quad \text{for } r = 0, 1, 2, 3.$$

Lemma 2.7 ([S1, Theorem 2.3]). *Let p be an odd prime, $r \in \{0, 1, 2, 3\}$, $k \in \mathbb{Z}_p$ and $k^2 + 1 \not\equiv 0 \pmod{p}$.*

(i) *If $p \equiv 1 \pmod{4}$ and $t^2 \equiv -1 \pmod{p}$ with $t \in \mathbb{Z}_p$, then $k \in Q_r(p)$ if and only if $(\frac{k+t}{k-t})^{(p-1)/4} \equiv t^r \pmod{p}$.*

(ii) *If $p \equiv 3 \pmod{4}$, then $k \in Q_r(p)$ if and only if $(\frac{k-i}{k+i})^{(p+1)/4} \equiv i^r \pmod{p}$.*

Lemma 2.8. *Let p be an odd prime, $k \in \mathbb{Z}_p$ and $n^2 \equiv k^2 + 1 \pmod{p}$ with $n \in \mathbb{Z}_p$ and $n(n+1) \not\equiv 0 \pmod{p}$. Then $(\frac{k+i}{p})_4 = (\frac{n(n+1)}{p})$.*

Proof. For $k \equiv 0 \pmod{p}$ we have $(\frac{k+i}{p})_4 = (\frac{i}{p})_4 = (-1)^{\frac{p^2-1}{8}} = (\frac{1 \cdot 2}{p})$. So the result is true. Now assume $k \not\equiv 0 \pmod{p}$. Then $(\frac{n-1}{p})(\frac{n+1}{p}) = (\frac{n^2-1}{p}) = (\frac{k^2}{p}) = 1$ and so $(\frac{n-1}{p}) = (\frac{n+1}{p})$. By Lemma 2.4, $(\frac{k+i}{p})_4^2 = (\frac{k^2+1}{p}) = 1$ and so $(\frac{k+i}{p})_4 = \pm 1$. By [S1, Theorem 2.4], $(\frac{k+i}{p})_4 = 1 \iff k \in Q_0(p) \iff (\frac{n(n+1)}{p}) = 1$. Hence $(\frac{k+i}{p})_4 = (\frac{n(n+1)}{p})$.

Lemma 2.9. *Suppose $c, d, m, x \in \mathbb{Z}$, $2 \nmid m$, $x^2 \equiv c^2 + d^2 \pmod{m}$ and $(m, x(x+d)) = 1$. Then*

$$\left(\frac{c+di}{m}\right)_4 = \left(\frac{x(x+d)}{m}\right).$$

Proof. Suppose that p is a prime divisor of m . Then $p \nmid x(x+d)$. If $p \nmid d$, then $(\frac{x}{d})^2 \equiv (\frac{c}{d})^2 + 1 \pmod{p}$. Thus, applying Lemma 2.8 we obtain

$$\left(\frac{c+di}{p}\right)_4 = \left(\frac{\frac{c}{d}+i}{p}\right)_4 = \left(\frac{\frac{x}{d}(1+\frac{x}{d})}{p}\right) = \left(\frac{x(x+d)}{p}\right).$$

When $p \mid d$, we have $p \nmid c$ and so

$$\left(\frac{c+di}{p}\right)_4 = \left(\frac{c}{p}\right)_4 = 1 = \left(\frac{x^2}{p}\right) = \left(\frac{x(x+d)}{p}\right).$$

Hence,

$$\left(\frac{c+di}{m}\right)_4 = \prod_{p|m} \left(\frac{c+di}{p}\right)_4 = \prod_{p|m} \left(\frac{x(x+d)}{p}\right) = \left(\frac{x(x+d)}{m}\right),$$

where in the products p runs over all prime divisors of m . The proof is now complete.

Lemma 2.10. *Suppose $c, d, x, y, q \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $2 \mid d$, $c^2 + d^2 = x^2 + qy^2$, $y = 2^t y_0$, $y_0 \equiv 1 \pmod{4}$ and $(y_0, x(x+d)) = 1$.*

(i) *If $2 \mid x$, then $(\frac{y}{c^2+(x+d)^2}) = (-1)^{\frac{y-1}{4}} (\frac{y^{-1}}{c+di})_4$.*

(ii) *If $2 \nmid x$, then $(\frac{y}{(c^2+(x+d)^2)/2}) = (-1)^{\frac{c^2-(x+d)^2}{8}t} i^{\frac{d}{2}t} (\frac{y^{-1}}{c+di})_4$.*

Proof. Since $c^2 + (x+d)^2 = 2x(x+d) + qy^2$ we see that $(c^2 + (x+d)^2, y_0) = 1$. For even x we have $2 \nmid qy$, $c^2 + (x+d)^2 \equiv 1 \pmod{4}$ and so

$$\left(\frac{y}{c^2+(x+d)^2}\right) = \left(\frac{c^2+(x+d)^2}{y}\right) = \left(\frac{2x(x+d)}{y}\right) = (-1)^{\frac{y-1}{4}} \left(\frac{x(x+d)}{y}\right).$$

For odd x we have $c^2 + (x + d)^2 \equiv 2 \pmod{8}$ and so

$$\begin{aligned} & \left(\frac{y}{(c^2 + (x + d)^2)/2} \right) \\ &= \left(\frac{2^t y_0}{(c^2 + (x + d)^2)/2} \right) = (-1)^{\frac{(c^2 + (x + d)^2)/2 - 1}{4} t} \left(\frac{(c^2 + (x + d)^2)/2}{y_0} \right) \\ &= (-1)^{\frac{c^2 - (x + d)^2}{8} t} \left(\frac{2}{y_0} \right) \left(\frac{2x(x + d) + qy^2}{y_0} \right) = (-1)^{\frac{c^2 - (x + d)^2}{8} t} \left(\frac{x(x + d)}{y_0} \right). \end{aligned}$$

Since $x^2 \equiv c^2 + d^2 \pmod{|y_0|}$, using Lemmas 2.9, 2.3 and 2.2 we see that

$$\left(\frac{x(x + d)}{y_0} \right) = \left(\frac{c + di}{y_0} \right)_4 = \left(\frac{y_0}{c + di} \right)_4 = \left(\frac{y_0^{-1}}{c + di} \right)_4 = \left(\frac{2^t y^{-1}}{c + di} \right)_4 = i^{\frac{d}{2} t} \left(\frac{y^{-1}}{c + di} \right)_4.$$

Now combining all the above we obtain the result.

Lemma 2.11. *Let p be a prime of the form $4k + 1$ and $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$. Suppose $q \in \mathbb{Z}$, $p \nmid q$ and $p = x^2 + qy^2$ with $x, y \in \mathbb{Z}$. Then $(x + d, c^2) = (x + d, qy^2)$ and*

$$(qy^2, c^2 + (x + d)^2) = (x + d, c^2) \left(2, x + d + \frac{c^2}{(x + d, c^2)} \right).$$

Proof. Since $(x, y)^2 \mid p$ we see that $(x, y) = 1$. If $p \mid x$, then $p \mid qy^2$ and so $p \mid y$. This contradicts the fact $(x, y) = 1$. Hence $p \nmid x$. Since $(x, c^2 + (x + d)^2) = (x, c^2 + d^2) = (x, p) = 1$ and $qy^2 = d^2 - x^2 + c^2 = c^2 + (x + d)^2 - 2x(x + d)$, we see that $(x + d, c^2) = (x + d, x^2 - d^2 + qy^2) = (x + d, qy^2)$ and

$$\begin{aligned} (qy^2, c^2 + (x + d)^2) &= (2x(x + d), c^2 + (x + d)^2) = (2(x + d), c^2 + (x + d)^2) \\ &= (x + d, c^2) \left(2 \frac{x + d}{(x + d, c^2)}, \frac{c^2}{(x + d, c^2)} + (x + d) \frac{x + d}{(x + d, c^2)} \right) \\ &= (x + d, c^2) \left(2, \frac{c^2}{(x + d, c^2)} + (x + d) \frac{x + d}{(x + d, c^2)} \right) \\ &= (x + d, c^2) \left(2, \frac{c^2}{(x + d, c^2)} + (x + d) \right). \end{aligned}$$

Thus the lemma is proved.

Lemma 2.12. *Let p be a prime of the form $4k + 1$ and $p = c^2 + d^2$ with $c, d \in \mathbb{Z}$ and $2 \nmid c$. Suppose $q \in \mathbb{Z}$, $p \nmid q$, $p = x^2 + qy^2$, $x, y \in \mathbb{Z}$ and $\left(\frac{x/y}{c+di} \right)_4 = (-1)^{\lfloor \frac{p}{8} \rfloor + n} i^k$. Then*

$$q^{[p/8]} \equiv \begin{cases} (-1)^n \left(\frac{d}{c} \right)^k \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^n \left(\frac{d}{c} \right)^k \frac{y}{x} \pmod{p} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Proof. It is clear that $(c, d) = 1$, $p \nmid y$ and so

$$\left(\frac{x}{y} \right)^{\frac{p-1}{4}} \equiv \left(\frac{x/y}{c + di} \right)_4 = (-1)^{\lfloor \frac{p}{8} \rfloor + n} i^k \equiv (-1)^{\lfloor \frac{p}{8} \rfloor + n} \left(\frac{d}{c} \right)^k \pmod{c + di}.$$

Thus $\left(\frac{x}{y}\right)^{\frac{p-1}{4}} \equiv (-1)^{\lfloor \frac{p}{8} \rfloor + n} \left(\frac{d}{c}\right)^k \pmod{p}$ and so

$$q^{\lfloor \frac{p}{8} \rfloor} = (-1)^{\lfloor \frac{p}{8} \rfloor} (-q)^{\lfloor \frac{p}{8} \rfloor} \equiv (-1)^{\lfloor \frac{p}{8} \rfloor} \left(\frac{x}{y}\right)^{2\lfloor \frac{p}{8} \rfloor} \equiv \begin{cases} (-1)^n \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } 8 \mid p-1, \\ (-1)^n \left(\frac{d}{c}\right)^k \frac{y}{x} \pmod{p} & \text{if } 8 \mid p-5. \end{cases}$$

This proves the lemma.

3. Determination of $q^{\lfloor p/8 \rfloor} \pmod{p}$ using $\left(\frac{c+(x+d)i}{q}\right)_4$ or $\left(\frac{d-(x+c)i}{q}\right)_4$.

Theorem 3.1. *Let p be a prime of the form $4k+1$, $q \in \mathbb{Z}$, $2 \nmid q$ and $p \nmid q$. Suppose that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $d_0 \equiv 1 \pmod{4}$, $(c, x+d) = 1$, $2 \mid x$, $y \equiv 1 \pmod{4}$ and $\left(\frac{c/(x+d)+i}{q}\right)_4 = i^k$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8} + \frac{d}{4} + \frac{x}{4}} \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{q-5}{8} + \frac{d}{4} + \frac{x-2}{4}} \left(\frac{d}{c}\right)^{k+1} \pmod{p} & \text{if } q \equiv 5 \pmod{8}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8} + \frac{x-2}{4}} \left(\frac{d}{c}\right)^{k+1} \frac{y}{x} \pmod{p} & \text{if } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{q-5}{8} + \frac{x}{4}} \left(\frac{d}{c}\right)^k \frac{y}{x} \pmod{p} & \text{if } q \equiv 5 \pmod{8}. \end{cases}$$

Proof. Suppose $2^m \parallel (x+d)$ and $x = 2^s x_0 (2 \nmid x_0)$. Since $2 \mid x$ we have $q \equiv 1 \pmod{4}$. As $(c, x+d) = 1$, by Lemma 2.11 we have $(qy, x+d) = 1$ and $(qy^2, c^2 + (x+d)^2) = 1$. Note that $(x, y)^2 \mid p$. We also have $(x, y) = 1$. Using Lemmas 2.1-2.5, 2.10 and the fact that $\left(\frac{a}{n}\right)_4 = 1$ for $a, n \in \mathbb{Z}$ with $2 \nmid n$ and $(a, n) = 1$ we see that

$$\begin{aligned} i^k &= \left(\frac{c+(x+d)i}{q}\right)_4 = \left(\frac{qy^2}{c+(x+d)i}\right)_4 \left(\frac{y^2}{c+(x+d)i}\right)_4 \\ &= \left(\frac{-2x(x+d) + c^2 + (x+d)^2}{c+(x+d)i}\right)_4 \left(\frac{y}{c^2 + (x+d)^2}\right)_4 \\ &= \left(\frac{-2x(x+d)}{c+(x+d)i}\right)_4 (-1)^{\frac{y-1}{4}} \left(\frac{y^{-1}}{c+di}\right)_4 \end{aligned}$$

and

$$\begin{aligned} \left(\frac{-2x(x+d)}{c+(x+d)i}\right)_4 &= \left(\frac{2}{c+(x+d)i}\right)_4^{m+s+1} \left(\frac{-x_0(x+d)/2^m}{c+(x+d)i}\right)_4 \\ &= i^{\frac{x+d}{2}(m+s+1)} (-1)^{\frac{x_0(x+d)/2^m+1}{2} \cdot \frac{x+d}{2}} \left(\frac{c+(x+d)i}{x_0(x+d)/2^m}\right)_4 \\ &= (-1)^{\frac{x_0(x+d)/2^m+1}{2} \cdot \frac{x+d}{2}} i^{\frac{x+d}{2}(m+s+1)} \left(\frac{c+di}{x_0}\right)_4 \left(\frac{c}{(x+d)/2^m}\right)_4 \\ &= (-1)^{\frac{x_0(x+d)/2^m+1}{2} \cdot \frac{x+d}{2}} i^{\frac{x+d}{2}(m+s+1)} (-1)^{\frac{x_0-1}{2} \cdot \frac{d}{2}} \left(\frac{x_0}{c+di}\right)_4 \\ &= (-1)^{\frac{x_0(x+d)/2^m+1}{2} \cdot \frac{x+d}{2}} i^{\frac{x+d}{2}(m+s+1)} (-1)^{\frac{x_0-1}{2} \cdot \frac{d}{2}} \left(\frac{2}{c+di}\right)_4^{-s} \left(\frac{x}{c+di}\right)_4 \\ &= (-1)^{\frac{x_0(x+d)/2^m+1}{2} \cdot \frac{x+d}{2}} i^{\frac{x+d}{2}(m+s+1)} (-1)^{\frac{x_0-1}{2} \cdot \frac{d}{2}} i^{-\frac{d}{2}s} \left(\frac{x}{c+di}\right)_4. \end{aligned}$$

Therefore,

$$i^k = (-1)^{\frac{x_0(x+d)/2^{m+1} \cdot \frac{x+d}{2} + \frac{x_0-1}{2} \cdot \frac{d}{2} + \frac{y-1}{4}} i^{\frac{x+d}{2}(m+s+1) - \frac{d}{2}s} \left(\frac{x/y}{c+di} \right)_4.$$

Observe that

$$(-1)^{\frac{y-1}{4}} = (-1)^{\frac{q(y^2-1)}{8}} = (-1)^{\frac{p-q-x^2}{8}} = \begin{cases} (-1)^{\frac{p-q-4}{8}} & \text{if } 2 \parallel x, \\ (-1)^{\frac{p-q}{8}} & \text{if } 4 \mid x \end{cases}$$

and

$$\begin{aligned} i^{\frac{d}{2}s - \frac{x+d}{2}(m+s+1)} &= i^{-\frac{x+d}{2}(m+1) - \frac{x}{2}s} = (-1)^{\frac{(m+1)(x+d)}{4}} i^{-\frac{x}{2}s} \\ &= \begin{cases} (-1)^{\frac{(m+1)(x+d)}{4}} \cdot (-1)^{\frac{x+2}{4}} i & \text{if } 2 \parallel x, \\ (-1)^{\frac{(m+1)(x+d)}{4}} & \text{if } 4 \mid x. \end{cases} \end{aligned}$$

From the above we obtain

$$(3.1) \quad \left(\frac{x/y}{c+di} \right)_4 = \begin{cases} (-1)^{\frac{x_0(x+d)/2^{m+1} \cdot \frac{x+d}{2} + \frac{x-2}{4} \cdot \frac{d}{2} + \frac{p-q-4}{8}} \cdot (-1)^{\frac{(m+1)(x+d)}{4} + \frac{x+2}{4}} i^{k+1} & \text{if } 2 \parallel x, \\ (-1)^{\frac{x_0(x+d)/2^{m+1} \cdot \frac{x+d}{2} + \frac{x_0-1}{2} \cdot \frac{d}{2} + \frac{p-q}{8}} \cdot (-1)^{\frac{(m+1)(x+d)}{4}} i^k & \text{if } 4 \mid x. \end{cases}$$

When $4 \mid (x+d)$, we have $(-1)^{\frac{(m+1)(x+d)}{4}} = (-1)^{\frac{x+d}{4}}$. For $p \equiv q \equiv 1 \pmod{8}$ we have $4 \mid d$, $4 \mid x$ and $4 \mid (x+d)$. For $p \equiv 1 \pmod{8}$ and $q \equiv 5 \pmod{8}$, we have $2 \parallel x$, $4 \mid d$, $2 \parallel (x+d)$, $m = 1$ and $(-1)^{\frac{x_0(x+d)/2^{2+1}}{2}} = (-1)^{\frac{d}{4}x_0 + \frac{x_0^2+1}{2}} = (-1)^{\frac{d}{4}+1}$. For $p \equiv 5 \pmod{8}$ and $q \equiv 1 \pmod{8}$, we have $2 \parallel d$, $2 \parallel x$, $4 \mid (x+d)$ and $m \geq 2$. For $p \equiv q \equiv 5 \pmod{8}$, we see that $2 \parallel d$, $4 \mid x$, $2 \parallel (x+d)$, $m = 1$ and $(-1)^{\frac{x_0(x+d)/2^{2+1}}{2}} = (-1)^{\frac{x}{4}x_0 + \frac{d_0x_0+1}{2}} = (-1)^{\frac{x}{4} + \frac{x_0+1}{2}}$. Now, from the above and (3.1) we deduce that

$$\left(\frac{x/y}{c+di} \right)_4 = \begin{cases} (-1)^{\frac{p-1}{8} + \frac{q-1}{8} + \frac{d}{4} + \frac{x}{4}} i^k & \text{if } p \equiv q \equiv 1 \pmod{8}, \\ (-1)^{\frac{p-1}{8} + \frac{q-5}{8} + \frac{d}{4} + \frac{x-2}{4}} i^{k+1} & \text{if } p \equiv 1 \pmod{8} \text{ and } q \equiv 5 \pmod{8}, \\ (-1)^{\frac{p-5}{8} + \frac{q-1}{8} + \frac{x-2}{4}} i^{k+1} & \text{if } p \equiv 5 \pmod{8} \text{ and } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{p-5}{8} + \frac{q-5}{8} + \frac{x}{4}} i^k & \text{if } p \equiv q \equiv 5 \pmod{8}. \end{cases}$$

This together with Lemma 2.12 yields the result.

Lemma 3.1. *Let p be a prime of the form $4k+1$, $q \in \mathbb{Z}$, $2 \nmid q$ and $p \nmid q$. Suppose that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$, $d_0 \equiv y_0 \equiv 1 \pmod{4}$, $(c, x+d) = 1$ and $2 \nmid x$. Assume that $(\frac{c/(x+d)+i}{q})_4 = i^k$. Then*

$$\left(\frac{x/y}{c+di} \right)_4 = \begin{cases} (-1)^{\lfloor \frac{p}{8} \rfloor + \frac{q - (\frac{-1}{q})}{4} \cdot \frac{x-1}{2} + \frac{q-1}{2} \cdot \frac{d}{4}} i^{\frac{1 - (\frac{-1}{q})q}{4} + \frac{x-1}{4} x-c + k} & \text{if } 8 \mid (p-1), \\ (-1)^{\lfloor \frac{p}{8} \rfloor + \frac{q+1}{2} + \frac{x+1}{2} (1 + \frac{q - (\frac{-1}{q})}{4})} i^{\frac{1 - (\frac{-1}{q})q}{4} + \frac{x-1}{4} x-c + k-1} & \text{if } 8 \mid (p-5). \end{cases}$$

Proof. As $(c, x+d) = 1$, by Lemma 2.11 we have $(qy_0, x+d) = 1$ and $(qy_0^2, c^2 + (x+d)^2) = 1$. Note that $(x, y)^2 \mid p$. We also have $(x, y) = 1$. It is easily seen that

$$c + (x+d)i = i^{\frac{1+\varepsilon}{2}} (1+i) \left(\frac{x+d \pm c}{2} + \frac{\pm(x+d) - c}{2} i \right)$$

and so

$$\left(\frac{x+d \pm c}{2} \right)^2 + \left(\frac{\pm(x+d) - c}{2} \right)^2 = \frac{c^2 + (x+d)^2}{2}.$$

Set $\varepsilon = (-1)^{\frac{p-1}{4} + \frac{x-1}{2}}$. Since $4 \mid d \iff 8 \mid (p-1)$ we see that $x+d \equiv \varepsilon \pmod{4}$ and $4 \mid (\varepsilon(x+d) - c)$. Using Lemmas 2.1-2.5, 2.10 and the above we see that

$$\begin{aligned} i^k &= \left(\frac{c + (x+d)i}{q} \right)_4 = \left(\frac{i}{q} \right)_4^{\frac{1-\varepsilon}{2}} \left(\frac{1+i}{q} \right)_4 \left(\frac{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i}{q} \right)_4 \\ &= (-1)^{\frac{q - (\frac{-1}{q})}{4} \cdot \frac{1-\varepsilon}{2}} i^{\frac{(\frac{-1}{q})^{q-1}}{4}} (-1)^{\frac{q-1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} \left(\frac{q}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \end{aligned}$$

and

$$\begin{aligned} &\left(\frac{q}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \\ &= \left(\frac{qy^2}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \left(\frac{y^2}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \\ &= \left(\frac{c^2 + (x+d)^2 - 2x(x+d)}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \left(\frac{y}{(\frac{x+d+\varepsilon c}{2})^2 + (\frac{\varepsilon(x+d)-c}{2})^2} \right) \\ &= \left(\frac{2}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \left(\frac{-x(x+d)}{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i} \right)_4 \left(\frac{y}{(c^2 + (x+d)^2)/2} \right) \\ &= i^{(-1)^{\frac{(x+d+\varepsilon c)/2-1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}}} (-1)^{\frac{x(x+d)+1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} \left(\frac{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i}{x(x+d)} \right)_4 \\ &\quad \times (-1)^{\frac{c^2 - (x+d)^2}{8}} t_i^{\frac{d}{2}t} \left(\frac{y^{-1}}{c+di} \right)_4. \end{aligned}$$

Obviously $i^{(-1)^a b} = (-1)^{ab} i^b$, $(-1)^{\frac{x+d+\varepsilon c-2}{4}} = (-1)^{\frac{\varepsilon(x+d)+c-2\varepsilon}{4}} = (-1)^{\frac{\varepsilon(x+d)-c}{4} + \frac{1-\varepsilon}{2}}$ and so

$$\begin{aligned} i^{(-1)^{\frac{x+d+\varepsilon c-2}{4} \cdot \frac{\varepsilon(x+d)-c}{4}}} &= (-1)^{\left(\frac{\varepsilon(x+d)-c}{4} + \frac{1-\varepsilon}{2}\right) \frac{\varepsilon(x+d)-c}{4}} i^{\frac{\varepsilon(x+d)-c}{4}} \\ &= (-1)^{\frac{1+\varepsilon}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} i^{\frac{\varepsilon(x+d)-c}{4}}. \end{aligned}$$

Also, $(-1)^{\frac{x(x+d)+1}{2}} = (-1)^{\frac{x^2-1}{2} + \frac{d}{2}x+1} = (-1)^{\frac{d}{2}+1}$ and $(-1)^{\frac{c^2 - (x+d)^2}{8}} = (-1)^{\frac{\varepsilon(x+d)-c}{4}}$. Thus,

$$\begin{aligned} &i^{(-1)^{\frac{(x+d+\varepsilon c)/2-1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}}} (-1)^{\frac{x(x+d)+1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} \cdot (-1)^{\frac{c^2 - (x+d)^2}{8}} t_i^{\frac{d}{2}t} \\ &= (-1)^{\frac{1+\varepsilon}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} i^{\frac{\varepsilon(x+d)-c}{4}} (-1)^{\left(\frac{d}{2}+1\right) \frac{\varepsilon(x+d)-c}{4}} \cdot (-1)^{\frac{\varepsilon(x+d)-c}{4}} t_i^{\frac{d}{2}t} \\ &= (-1)^{\left(\frac{1-\varepsilon}{2} + \frac{d}{2} + t\right) \frac{\varepsilon(x+d)-c}{4}} i^{\frac{\varepsilon(x+d)-c}{4} + \frac{d}{2}t}. \end{aligned}$$

It is easily seen that

$$\begin{aligned}
& \left(\frac{\frac{x+d+\varepsilon c}{2} + \frac{\varepsilon(x+d)-c}{2} i}{x(x+d)} \right)_4 \\
&= \left(\frac{x+d+\varepsilon c + (\varepsilon(x+d)-c)i}{x} \right)_4 \left(\frac{x+d+\varepsilon c + (\varepsilon(x+d)-c)i}{x+d} \right)_4 \\
&= \left(\frac{d+\varepsilon c + (\varepsilon d-c)i}{x} \right)_4 \left(\frac{\varepsilon c - ci}{x+d} \right)_4 = \left(\frac{(\varepsilon-i)(c+di)}{x} \right)_4 \left(\frac{\varepsilon-i}{x+d} \right)_4 \\
&= \left(\frac{\varepsilon-i}{x(x+d)} \right)_4 \left(\frac{c+di}{x} \right)_4 = \left(\frac{i^{\frac{5+\varepsilon}{2}}(1+i)}{x(x+d)} \right)_4 \left(\frac{c+di}{x} \right)_4 \\
&= \left(\frac{i}{x(x+d)} \right)_4^{\frac{5+\varepsilon}{2}} \left(\frac{1+i}{x(x+d)} \right)_4 (-1)^{\frac{x-1}{2} \cdot \frac{d}{2}} \left(\frac{x}{c+di} \right)_4 \\
&= (-1)^{\frac{(x(d+x))^2-1}{8} \cdot \frac{5+\varepsilon}{2}} i^{\frac{(-1)^{d/2}x(x+d)-1}{4}} (-1)^{\frac{x-1}{2} \cdot \frac{d}{2}} \left(\frac{x}{c+di} \right)_4.
\end{aligned}$$

Now combining all the above we deduce that

$$\begin{aligned}
(3.2) \quad i^k &= (-1)^{\frac{q-\frac{(-1)}{q}}{4} \cdot \frac{1-\varepsilon}{2} + \frac{q-1}{2} \cdot \frac{\varepsilon(x+d)-c}{4}} i^{\frac{(-1)}{q}q-1} \\
&\quad \times (-1)^{\left(\frac{1-\varepsilon}{2} + \frac{d}{2} + t\right) \frac{\varepsilon(x+d)-c}{4}} i^{\frac{\varepsilon(x+d)-c}{4} + \frac{d}{2}t} \\
&\quad \times (-1)^{\frac{(x(d+x))^2-1}{8} \cdot \frac{5+\varepsilon}{2} + \frac{x-1}{2} \cdot \frac{d}{2}} i^{\frac{(-1)^{d/2}x(x+d)-1}{4}} \left(\frac{x/y}{c+di} \right)_4.
\end{aligned}$$

It is clear that

$$\begin{aligned}
(-1)^{\frac{\varepsilon(x+d)-c}{4}} &= (-1)^{\frac{\varepsilon(x+d)-c}{4} \cdot \frac{\varepsilon(x+d)+c}{2}} = (-1)^{\frac{(x+d)^2-c^2}{8}} = (-1)^{\frac{2d^2+2dx-xy^2}{8}} \\
&= \begin{cases} (-1)^{\frac{d}{4}} & \text{if } 8 \mid (p-1) \text{ and so } d \equiv y \equiv 0 \pmod{4}, \\ (-1)^{\frac{2+x-q}{2}} = (-1)^{\frac{q-1}{2} + \frac{1-\varepsilon}{2}} & \text{if } 8 \mid (p-5) \text{ and so } d \equiv y \equiv 2 \pmod{4} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(-1)^{\frac{x^2(x+d)^2-1}{8}} &= (-1)^{\frac{(-1)^{d/2}x(x+d)-1}{4}} = (-1)^{\frac{(-1)^{d/2}(dx+1)-1}{4}} \\
&= \begin{cases} (-1)^{\frac{d}{4}} & \text{if } 8 \mid (p-1) \text{ and so } 4 \mid d, \\ (-1)^{\frac{d_0x+1}{2}} = \varepsilon & \text{if } 8 \mid (p-5) \text{ and so } d \equiv y \equiv 2 \pmod{4}. \end{cases}
\end{aligned}$$

Since $x+d \equiv \varepsilon \pmod{4}$ and $(-1)^{\frac{d}{2}} = (-1)^{\frac{p-1}{4}}$ we also have

$$\begin{aligned}
& i^{\frac{\varepsilon(x+d)-c}{4} + \frac{(-1)^{d/2}x(x+d)-1}{4} + \frac{d}{2}t} \\
&= (-1)^{\frac{(-1)^{d/2}x(x+d)-1}{4}} i^{\frac{\varepsilon(x+d)-c}{4} - \frac{(-1)^{(p-1)/4}x(x+d)-1}{4} + \frac{d}{2}t} \\
&= (-1)^{\frac{(-1)^{d/2}x(x+d)-1}{4}} i^{\varepsilon(x+d) \frac{1-(-1)^{(x-1)/2}x - \frac{c-1}{4}}{4} + \frac{d}{2}t} \\
&= (-1)^{\frac{(-1)^{d/2}x(x+d)-1}{4} + \frac{c-1}{4}} i^{\frac{1-(-1)^{(x-1)/2}x + \frac{c-1}{4}}{4} + \frac{d}{2}t} \\
&= \begin{cases} (-1)^{\frac{d}{4} + \frac{c-1}{4} + \frac{d}{4}t} i^{\frac{c-(-1)^{(x-1)/2}x}{4}} & \text{if } 8 \mid (p-1), \\ \varepsilon (-1)^{\frac{c-1}{4}} i^{\frac{c-(-1)^{(x-1)/2}x}{4} + d_0} = (-1)^{\frac{1-\varepsilon}{2} + \frac{c-1}{4}} i^{\frac{c-(-1)^{(x-1)/2}x}{4} + 1} & \text{if } 8 \mid (p-5). \end{cases}
\end{aligned}$$

Note that $(-1)^{\frac{c-1}{4}} = (-1)^{\frac{c^2-1}{8}} = (-1)^{\frac{p-1-d^2}{8}} = (-1)^{\lfloor \frac{p}{8} \rfloor}$. From the above and (3.2) we deduce the result.

Theorem 3.2. *Let p be a prime of the form $4k+1$, $q \in \mathbb{Z}$, $2 \nmid q$ and $p \nmid q$. Suppose that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$, $d_0 \equiv y_0 \equiv 1 \pmod{4}$, $(c, x+d) = 1$, $2 \nmid x$ and $(\frac{c/(x+d)+i}{q})_4 = i^k$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8} + \frac{d}{4} + \frac{y}{4}} \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{q+5}{8} + \frac{x-1}{2} + \frac{y}{4}} \left(\frac{d}{c}\right)^{k-1} \pmod{p} & \text{if } q \equiv 3 \pmod{8}, \\ (-1)^{\frac{q-5}{8} + \frac{d}{4} + \frac{x-1}{2} + \frac{y}{4}} \left(\frac{d}{c}\right)^{k-1} \pmod{p} & \text{if } q \equiv 5 \pmod{8}, \\ (-1)^{\frac{q+1}{8} + \frac{y}{4}} \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } q \equiv 7 \pmod{8}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8} + \frac{x-1}{2}} \left(\frac{d}{c}\right)^{k-1} \frac{y}{x} \pmod{p} & \text{if } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{q+5}{8}} \left(\frac{d}{c}\right)^{k-1} \frac{y}{x} \pmod{p} & \text{if } q \equiv 3 \pmod{8}, \\ (-1)^{\frac{q+3}{8}} \left(\frac{d}{c}\right)^k \frac{y}{x} \pmod{p} & \text{if } q \equiv 5 \pmod{8}, \\ (-1)^{\frac{q+1}{8} + \frac{x-1}{2}} \left(\frac{d}{c}\right)^k \frac{y}{x} \pmod{p} & \text{if } q \equiv 7 \pmod{8}. \end{cases}$$

Proof. Suppose $c \neq (-1)^{(x-1)/2}x$ and $2^m \parallel (c - (-1)^{(x-1)/2}x)$. Then $m \geq 2$ and $2^{m+1} \parallel (c-x)(c+x)$. As $d^2 - qy^2 = -(c-x)(c+x)$ we see that $2^{m+1} \parallel (d^2 - qy^2)$. We first assume $p \equiv 1 \pmod{8}$. Since $4 \mid d$ and $4 \mid y$ we have $m \geq 3$ and $2^{m-3} \parallel ((\frac{d}{4})^2 - q(\frac{y}{4})^2)$. Thus,

$$i^{\frac{(-1)^{(x-1)/2}x-c}{4}} = (-1)^{\frac{(-1)^{(x-1)/2}x-c}{8}} = (-1)^{2^{m-3}} = (-1)^{(\frac{d}{4})^2 - q(\frac{y}{4})^2} = (-1)^{\frac{d}{4} + \frac{y}{4}}.$$

This is also true when $c = (-1)^{(x-1)/2}x$. From the above and Lemma 3.1 we deduce that

$$\left(\frac{x/y}{c+di}\right)_4 = \begin{cases} (-1)^{\frac{p-1}{8} + \frac{q-(-1)}{8} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} i^k & \text{if } q \equiv \pm 1 \pmod{8}, \\ (-1)^{\frac{p-1}{8} + \frac{q-5(-1)}{8} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{x-1}{2} + \frac{y}{4}} i^{k-1} & \text{if } q \equiv \pm 3 \pmod{8}. \end{cases}$$

Now applying Lemma 2.12 we deduce (i).

Suppose $p \equiv 5 \pmod{8}$. As $qy^2 - d^2 = 4(qy_0^2 - d_0^2)$ we get $2^{m-1} \parallel (qy_0^2 - d_0^2)$. Clearly $qy_0^2 - d_0^2 \equiv q - 1 \pmod{8}$. Thus

$$q \equiv \begin{cases} 3 \pmod{4} & \text{if } m = 2, \\ 5 \pmod{8} & \text{if } m = 3, \\ 1 \pmod{8} & \text{if } m > 3. \end{cases}$$

For $q \equiv 1 \pmod{4}$ we have $8 \mid (c - (-1)^{\frac{x-1}{2}}x)$ and

$$i^{\frac{(-1)^{(x-1)/2}x-c}{4}} = (-1)^{\frac{(-1)^{(x-1)/2}x-c}{8}} = (-1)^{2^{m-3}} = (-1)^{\frac{q-1}{4}}.$$

This is also true when $c = (-1)^{(x-1)/2}x$. Thus, using Lemma 3.1 we deduce that

$$\left(\frac{x/y}{c+di}\right)_4 = \begin{cases} (-1)^{\frac{p-5}{8} + \frac{q-1}{8} + \frac{x-1}{2}} i^{k-1} & \text{if } q \equiv 1 \pmod{8}, \\ (-1)^{\frac{p-5}{8} + \frac{q+3}{8}} i^k & \text{if } q \equiv 5 \pmod{8}. \end{cases}$$

Now applying Lemma 2.12 we deduce the result in the case $p \equiv 5 \pmod{8}$ and $q \equiv 1 \pmod{4}$. For $q \equiv 3 \pmod{4}$ we have $2^2 \parallel (c - (-1)^{(x-1)/2}x)$ and so

$$q \equiv qy_0^2 = 2c \cdot \frac{c - (-1)^{\frac{x-1}{2}}x}{4} - 4\left(\frac{c - (-1)^{\frac{x-1}{2}}x}{4}\right)^2 + d_0^2 \equiv 2 \cdot \frac{c - (-1)^{\frac{x-1}{2}}x}{4} - 4 + 1 \pmod{8}.$$

Thus, $\frac{(-1)^{(x-1)/2}x-c}{4} \equiv -\frac{q+3}{2} \pmod{4}$ and so $i^{\frac{(-1)^{(x-1)/2}x-c}{4}} = i^{-\frac{q+3}{2}}$. Using Lemma 3.1 we see that

$$\left(\frac{x/y}{c+di}\right)_4 = \begin{cases} (-1)^{\frac{p-5}{8} + \frac{q+5}{8}} i^{k-1} & \text{if } q \equiv 3 \pmod{8}, \\ (-1)^{\frac{p-5}{8} + \frac{q+1}{8} + \frac{x-1}{2}} i^k & \text{if } q \equiv 7 \pmod{8}. \end{cases}$$

Now applying Lemma 2.12 we obtain the result in the case $p \equiv 5 \pmod{8}$ and $q \equiv 3 \pmod{4}$.

Summarizing all the above we prove the theorem.

Remark 3.1 We note that the k in Theorems 3.1-3.2 depends only on $\frac{c}{x+d} \pmod{q}$.

Corollary 3.1. *Let $p \equiv 1, 49 \pmod{60}$ be a prime and so $p = c^2 + d^2 = x^2 + 15y^2$ with $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$, $d_0 \equiv y_0 \equiv 1 \pmod{4}$ and $(c, x+d) = 1$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$15^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{y}{4}} \pmod{p} & \text{if } \frac{c}{x+d} \equiv 0, \pm 1 \pmod{15}, \\ -(-1)^{\frac{y}{4}} \pmod{p} & \text{if } \frac{c}{x+d} \equiv \pm 4 \pmod{15}, \\ \pm(-1)^{\frac{y}{4}} \frac{c}{d} \pmod{p} & \text{if } \frac{c}{x+d} \equiv \pm 5, \pm 6 \pmod{15}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$15^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{x-1}{2}} \frac{y}{x} \pmod{p} & \text{if } \frac{c}{x+d} \equiv 0, \pm 1 \pmod{15}, \\ -(-1)^{\frac{x-1}{2}} \frac{y}{x} \pmod{p} & \text{if } \frac{c}{x+d} \equiv \pm 4 \pmod{15}, \\ \mp(-1)^{\frac{x-1}{2}} \frac{dy}{cx} \pmod{p} & \text{if } \frac{c}{x+d} \equiv \pm 5, \pm 6 \pmod{15}. \end{cases}$$

Proof. Clearly x is odd. Thus, putting $q = 15$ in Theorem 3.2 and noting that (see [S1, Example 2.1])

$$\left(\frac{n+i}{15}\right)_4 = \left(\frac{n+i}{3}\right)_4 \left(\frac{n+i}{5}\right)_4 = \begin{cases} 1 & \text{if } n \equiv 0, \pm 1 \pmod{15}, \\ -1 & \text{if } n \equiv \pm 4 \pmod{15}, \\ \mp i & \text{if } n \equiv \pm 5, \pm 6 \pmod{15} \end{cases}$$

we deduce the result.

For example, since $61 = 5^2 + (-6)^2 = (-1)^2 + 15 \cdot 2^2$, $(5, -1 - 6) = 1$ and $\frac{5}{-1-6} \equiv -5 \pmod{15}$ we have

$$15^{\frac{61-5}{8}} = 15^7 \equiv (-1)^{\frac{-1-1}{2}} \frac{-6 \cdot 2}{5 \cdot (-1)} \equiv -\frac{12}{5} \equiv 22 \pmod{61}.$$

Theorem 3.3. *Let p be a prime of the form $4k+1$, $q \in \mathbb{Z}$, $2 \nmid q$ and $p \nmid q$. Suppose that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$, $d_0 \equiv y_0 \equiv 1 \pmod{4}$, $(d_0, x+c) = 1$ and $\left(\frac{d/(x+c)-i}{q}\right)_4 = i^k$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{d}{4} + [\frac{x}{4}]} \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } 2 \mid x, \\ (-1)^{\frac{q^2-1}{8} \cdot \frac{x+1}{2} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} \left(\frac{d}{c}\right)^k \pmod{p} & \text{if } 2 \nmid x. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{[\frac{x+2}{4}]} \left(\frac{d}{c}\right)^{k-1} \frac{y}{x} \pmod{p} & \text{if } 2 \mid x \text{ and so } q \equiv 1 \pmod{4}, \\ -(-1)^{\frac{q+3}{4} \cdot \frac{x+1}{2}} \left(\frac{d}{c}\right)^{k-1} \frac{y}{x} \pmod{p} & \text{if } 2 \nmid x \text{ and } q \equiv 1 \pmod{4}, \\ -(-1)^{\frac{q-3}{4} \cdot \frac{x+1}{2}} \left(\frac{d}{c}\right)^k \frac{y}{x} \pmod{p} & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

Proof. Suppose $x = 2^s x_0$ ($2 \nmid x_0$) and $2^m \parallel (x+c)$. As $(x+c, d_0) = 1$, by Lemma 2.11 we have $(qy_0, x+c) = 1$ and $(qy_0^2, (x+c)^2 + d^2) = 1$. Note that $(x, y)^2 \mid p$. We also have $(x, y) = 1$. If $m < r$, using Lemmas 2.1-2.5 and the fact $\left(\frac{a}{q}\right)_4 = 1$ for $a \in \mathbb{Z}$ with $(a, q) = 1$ we see that

(3.3)

$$\begin{aligned} \left(\frac{d - (x+c)i}{q}\right)_4 &= \left(\frac{-2^m}{q}\right)_4 \left(\frac{i}{q}\right)_4 \left(\frac{\frac{x+c}{2^m} + \frac{d}{2^m}i}{q}\right)_4 = (-1)^{\frac{q^2-1}{8} + \frac{q-1}{2} \cdot \frac{d}{2^{m+1}}} \left(\frac{q}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \\ &= (-1)^{\frac{q^2-1}{8} + \frac{q-1}{2} \cdot \frac{d}{2^{m+1}}} \left(\frac{qy^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \left(\frac{y^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \\ &= (-1)^{\frac{q^2-1}{8} + \frac{q-1}{2} \cdot \frac{d}{2^{m+1}}} \left(\frac{c^2 + d^2 - x^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \left(\frac{y}{\frac{(x+c)^2 + d^2}{2^{2m}}}\right)_4. \end{aligned}$$

If $m = r$, then clearly

$$(3.4) \quad \left(\frac{d - (x+c)i}{q}\right)_4 = \left(\frac{-2^r}{q}\right)_4 \left(\frac{i}{q}\right)_4 \left(\frac{\frac{x+c}{2^r} + \frac{d}{2^r}i}{q}\right)_4 = \left(\frac{2}{q}\right) \left(\frac{1+i}{q}\right)_4 \left(\frac{\frac{x+c+d}{2^{r+1}} - \frac{x+c-d}{2^{r+1}}i}{q}\right)_4.$$

If $m > r$, using Lemmas 2.1-2.5 we see that

$$(3.5) \quad \begin{aligned} \left(\frac{d - (x+c)i}{q}\right)_4 &= \left(\frac{\frac{d}{2^r} - \frac{x+c}{2^r}i}{q}\right)_4 = (-1)^{\frac{q-1}{2} \cdot \frac{x+c}{2^{r+1}}} \left(\frac{q}{\frac{d}{2^r} - \frac{x+c}{2^r}i}\right)_4 \\ &= (-1)^{\frac{q-1}{2} \cdot \frac{x+c}{2^{r+1}}} \left(\frac{qy^2}{\frac{d}{2^r} - \frac{x+c}{2^r}i}\right)_4 \left(\frac{y^2}{\frac{d}{2^r} - \frac{x+c}{2^r}i}\right)_4 \\ &= (-1)^{\frac{q-1}{2} \cdot \frac{x+c}{2^{r+1}}} \left(\frac{(x+c)^2 + d^2 - 2x(x+c)}{\frac{d}{2^r} - \frac{x+c}{2^r}i}\right)_4 \left(\frac{y}{\left(\frac{d}{2^r}\right)^2 + \left(\frac{x+c}{2^r}\right)^2}\right)_4 \\ &= (-1)^{\frac{q-1}{2} \cdot 2^{m-r-1}} \left(\frac{-2x(x+c)}{\frac{d}{2^r} - \frac{x+c}{2^r}i}\right)_4 \left(\frac{y}{((x+c)^2 + d^2)/2^{2r}}\right)_4. \end{aligned}$$

By considering the three cases $m < r$, $m = r$ and $m > r$ and applying lemmas in Section 2, one may deduce the result after doing horrible long calculations. We only prove the result in the case $m < r$ (including $2 \mid x$ and $4 \mid x-1$). The remaining two cases can be proved similarly. For the details in the cases $m = r$ and $m > r$, see the author's fourth version of "Quartic, octic residues and binary quadratic forms" in arXiv:1108.3027.

Now suppose $m < r$. Then

$$\begin{aligned} &\left(\frac{c^2 + d^2 - x^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \\ &= \left(\frac{-2x(x+c) + d^2 + (x+c)^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 = \left(\frac{-2x(x+c)}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \\ &= \left(\frac{2^{m+s+1}}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \left(\frac{-x_0}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \left(\frac{(x+c)/2^m}{\frac{x+c}{2^m} + \frac{d}{2^m}i}\right)_4 \\ &= i^{(-1)^{\frac{x+c}{2^m}-1}/2} \frac{d}{2^{m+1}} (m+s+1) (-1)^{\left(\frac{x_0+1}{2} + \frac{x+c-1}{2}\right)} \frac{d}{2^{m+1}} \left(\frac{\frac{x+c}{2^m} + \frac{d}{2^m}i}{x_0}\right)_4 \left(\frac{\frac{x+c}{2^m} + \frac{d}{2^m}i}{\frac{x+c}{2^m}}\right)_4 \\ &= i^{(-1)^{\frac{x+c}{2^m}-1}/2} \frac{d}{2^{m+1}} (m+s+1) \cdot (-1)^{\frac{\frac{x+c}{2^m} + x_0}{2}} \frac{d}{2^{m+1}} \left(\frac{x+c+di}{x_0}\right)_4 \left(\frac{i}{\frac{x+c}{2^m}}\right)_4. \end{aligned}$$

Clearly $\left(\frac{i}{\frac{x+c}{2^m}}\right)_4 = \left(\frac{2}{\frac{x+c}{2^m}}\right)_4$ and

$$\begin{aligned} \left(\frac{x+c+di}{x_0}\right)_4 &= \left(\frac{c+di}{x_0}\right)_4 = (-1)^{\frac{d}{2} \cdot \frac{x_0-1}{2}} \left(\frac{x_0}{c+di}\right)_4 \\ &= (-1)^{\frac{d}{2} \cdot \frac{x_0-1}{2}} \left(\frac{2}{c+di}\right)_4^{-s} \left(\frac{x}{c+di}\right)_4 = (-1)^{\frac{d}{2} \cdot \frac{x_0-1}{2}} i^{-\frac{d}{2}s} \left(\frac{x}{c+di}\right)_4. \end{aligned}$$

Thus,

$$\begin{aligned} & \left(\frac{c^2 + d^2 - x^2}{\frac{x+c}{2^m} + \frac{d}{2^m}i} \right)_4 \\ &= (-1)^{\frac{\frac{x+c}{2^m} + x_0}{2} \cdot \frac{d}{2^{m+1}} + \frac{x_0-1}{2} \cdot \frac{d}{2}} i^{(-1)^{\frac{x+c}{2^m} - 1}/2} \frac{d}{2^{m+1}} (m+s+1) - \frac{d}{2} s \left(\frac{2}{\frac{x+c}{2^m}} \right) \left(\frac{x}{c+di} \right)_4. \end{aligned}$$

As $((x+c)^2 + d^2)/2^{2m} \equiv 1 + 4d/2^{m+1} \pmod{8}$ and $x^2 \equiv c^2 + d^2 \pmod{|y_0|}$, using Lemma 2.9 we see that

$$\begin{aligned} & \left(\frac{y}{((x+c)^2 + d^2)/2^{2m}} \right) \\ &= \left(\frac{2^t y_0}{((x+c)^2 + d^2)/2^{2m}} \right) = (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{(x+c)^2 + d^2}{y_0} \right) = (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{2x(x+c)}{y_0} \right) \\ &= (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{2}{y_0} \right) \left(\frac{-d+ci}{y_0} \right)_4 = (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{-i}{y_0} \right)_4 \left(\frac{-d+ci}{y_0} \right)_4 \\ &= (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{c+di}{y_0} \right)_4 = (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{y_0}{c+di} \right)_4 = (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{y_0^{-1}}{c+di} \right)_4 \\ &= (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{2^t y^{-1}}{c+di} \right)_4 = (-1)^{\frac{d}{2^{m+1}} t} i^{\frac{d}{2} t} \left(\frac{y^{-1}}{c+di} \right)_4. \end{aligned}$$

From the above and (3.3) we deduce that

$$(3.6) \quad \begin{aligned} \left(\frac{d - (x+c)i}{q} \right)_4 &= (-1)^{\frac{q^2-1}{8} + \frac{q-1}{2} \cdot \frac{d}{2^{m+1}}} \cdot (-1)^{\frac{\frac{x+c}{2^m} + x_0}{2} \cdot \frac{d}{2^{m+1}} + \frac{x_0-1}{2} \cdot \frac{d}{2}} \left(\frac{2}{\frac{x+c}{2^m}} \right) \\ &\quad \times i^{(-1)^{\frac{x+c}{2^m} - 1}/2} \frac{d}{2^{m+1}} (m+s+1) - \frac{d}{2} s (-1)^{\frac{d}{2^{m+1}} t} i^{\frac{d}{2} t} \left(\frac{x/y}{c+di} \right)_4. \end{aligned}$$

If $m = 0$, then $2 \nmid x+c$, $2 \mid x$, $2 \nmid y$ and so $q \equiv p \equiv 1 \pmod{4}$. Thus, from (3.6) we deduce that

$$\left(\frac{d - (x+c)i}{q} \right)_4 = (-1)^{\frac{q-1}{4} + (\frac{x}{2}+1)\frac{d}{2}} i^{\frac{d}{2}} \left(\frac{2}{x+c} \right) \left(\frac{x/y}{c+di} \right)_4.$$

Since $\left(\frac{d-(x+c)i}{q} \right)_4 = i^k$, $(-1)^{\frac{q-1}{4}} = (-1)^{\frac{qy^2-1}{4}} = (-1)^{\frac{c^2-1+d^2-x^2}{4}} = (-1)^{\frac{d^2-x^2}{4}} = (-1)^{\frac{x}{2} - \frac{d}{2}}$ and

$$\left(\frac{2}{x+c} \right) = (-1)^{\frac{(x+c)^2-1}{8}} = (-1)^{\frac{c^2-1}{8} + \frac{x(\frac{x}{2}+c)}{2}} = (-1)^{\frac{p-1-d^2}{8} + \frac{x(\frac{x}{2}+1)}{2}} = (-1)^{\lfloor \frac{p}{8} \rfloor + \lfloor \frac{x+2}{4} \rfloor},$$

from the above and the fact $d_0 \equiv 1 \pmod{4}$ we derive that

$$\begin{aligned} \left(\frac{x/y}{c+di} \right)_4 &= (-1)^{\frac{x}{2} - \frac{d}{2} + (\frac{x}{2}+1)\frac{d}{2}} \cdot (-1)^{\lfloor \frac{p}{8} \rfloor + \lfloor \frac{x+2}{4} \rfloor} i^{k - \frac{d}{2}} \\ &= \begin{cases} (-1)^{\lfloor \frac{p}{8} \rfloor + \frac{x}{2} - \lfloor \frac{x+2}{4} \rfloor + \frac{d}{4}} i^k = (-1)^{\lfloor \frac{p}{8} \rfloor + \frac{d}{4} + \lfloor \frac{x}{4} \rfloor} i^k & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\lfloor \frac{p}{8} \rfloor + \lfloor \frac{x+2}{4} \rfloor} i^{k-1} & \text{if } p \equiv 5 \pmod{8}. \end{cases} \end{aligned}$$

Now applying Lemma 2.12 we obtain the result in the case $m = 0$.

If $m = 1 < r$, then $x \equiv 1 \pmod{4}$, $s = 0$, $4 \mid d$ and $p \equiv 1 \pmod{8}$. Since $p \equiv x^2 \equiv 1 \pmod{8}$ we have $8 \mid qy^2$ and so $4 \mid y$. Thus,

$$\begin{aligned} 1 - c \cdot \frac{x+c}{2} &\equiv \left(\frac{x+c}{2}\right)^2 - c \cdot \frac{x+c}{2} = 4\left(\left(\frac{d}{4}\right)^2 - q\left(\frac{y}{4}\right)^2\right) \\ &\equiv 4\left(\frac{d}{4} + \frac{y}{4}\right) = d + y \pmod{8} \end{aligned}$$

and so $\frac{x+c}{2} \equiv c - d - y \pmod{8}$. Therefore $\frac{x+c}{2} \equiv 1 \pmod{4}$ and

$$\left(\frac{2}{\frac{x+c}{2}}\right) = \left(\frac{2}{c-d-y}\right) = (-1)^{\frac{c-1}{4} + \frac{d}{4} + \frac{y}{4}} = (-1)^{\frac{c^2-1}{8} + \frac{d}{4} + \frac{y}{4}} = (-1)^{\frac{p-1}{8} + \frac{d}{4} + \frac{y}{4}}.$$

Hence, from (3.6) we deduce that

$$\begin{aligned} \left(\frac{d-(x+c)i}{q}\right)_4 &= (-1)^{\frac{q^2-1}{8} + \frac{q-1}{2} \cdot \frac{d}{4}} \cdot (-1)^{\frac{x+c+x}{2} \cdot \frac{d}{4}} \cdot (-1)^{\frac{d}{4}} \cdot (-1)^{\frac{p-1}{8} + \frac{d}{4} + \frac{y}{4}} \cdot \left(\frac{x/y}{c+di}\right)_4 \\ &= (-1)^{\frac{p-1}{8} + \frac{q^2-1}{8} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} \left(\frac{x/y}{c+di}\right)_4. \end{aligned}$$

Since $\left(\frac{d-(x+c)i}{q}\right)_4 = i^k$, we get $\left(\frac{x/y}{c+di}\right)_4 = (-1)^{\frac{p-1}{8} + \frac{q^2-1}{8} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} i^k$. Now applying

Lemma 2.12 we obtain $q^{\frac{p-1}{8}} \equiv (-1)^{\frac{q^2-1}{8} + \frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} \left(\frac{d}{c}\right)^k \pmod{p}$ as asserted.

Now we assume $2 \leq m < r$. Then $x \equiv 3 \pmod{4}$, $s = 0$, $8 \mid d$ and so $p \equiv 1 \pmod{8}$. Since $qy^2 = d^2 - (x+c)^2 + 2c(x+c)$ we see that

$$q \frac{y^2}{2^{m+1}} = 2^{2r-m-1} d_0^2 - 2^{m-1} \left(\frac{x+c}{2^m}\right)^2 + c \cdot \frac{x+c}{2^m}.$$

As $m \geq 2$ and $2r \geq 2(m+1) \geq m+4$, we must have $2^{m+1} \parallel y^2$, $2 \mid m+1$, $t = \frac{m+1}{2}$ and $q \equiv -2^{m-1} + c \cdot \frac{x+c}{2^m} \pmod{8}$. Thus $m \geq 3$, $\frac{x+c}{2^m} \equiv c(2^{m-1} + q) \equiv c - 1 + 2^{m-1} + q \pmod{8}$ and so $\frac{x+c}{2^m} \equiv q \pmod{4}$. Therefore, by (3.6) we have

$$\begin{aligned} \left(\frac{d-(x+c)i}{q}\right)_4 &= (-1)^{\frac{q^2-1}{8} + \frac{d}{2^{m+1}} \cdot \frac{m+1}{2}} \left(\frac{2}{c-1+2^{m-1}+q}\right) (-1)^{\frac{d}{2^{m+1}} t} \left(\frac{x/y}{c+di}\right)_4 \\ &= (-1)^{\frac{c-1}{4} + 2^{m-3}} \left(\frac{x/y}{c+di}\right)_4. \end{aligned}$$

Note that $(-1)^{\frac{c-1}{4}} = (-1)^{\frac{c^2-1}{8}} = (-1)^{\frac{p-1}{8}}$, $\left(\frac{d-(x+c)i}{q}\right)_4 = i^k$ and

$$(-1)^{2^{m-3}} = \begin{cases} 1 & \text{if } m > 3, \\ -1 & \text{if } m = 3 \end{cases} = \begin{cases} 1 & \text{if } t > 2, \\ -1 & \text{if } t = 2 \end{cases} = (-1)^{\frac{y}{4}}.$$

We then get $\left(\frac{x/y}{c+di}\right)_4 = (-1)^{\frac{p-1}{8} + \frac{y}{4}} i^k$. Recall that $8 \mid d$. Applying Lemma 2.12 we obtain $q^{\frac{p-1}{8}} \equiv (-1)^{\frac{y}{4}} \left(\frac{d}{c}\right)^k = (-1)^{\frac{q+1}{2} \cdot \frac{d}{4} + \frac{y}{4}} \left(\frac{d}{c}\right)^k \pmod{p}$. This proves the result in the case $2 \leq m < r$.

Summarizing all the above we prove the theorem.

4. New reciprocity laws for quartic and octic residues.

Theorem 4.1. *Let p and q be primes such that $p \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$. Suppose $p = c^2 + d^2 = x^2 + qy^2$, $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$, $d_0 \equiv y_0 \equiv 1 \pmod{4}$ and $(\frac{c-di}{x})^{\frac{q+1}{4}} \equiv i^m \pmod{q}$. Assume $(c, x+d) = 1$ or $(d_0, x+c) = 1$. Then*

$$q^{[p/8]} \equiv \begin{cases} (-1)^{\frac{y}{4} + \frac{q+1}{4} \cdot \frac{x-1}{2}} \left(\frac{d}{c}\right)^m \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{q-3}{4} \cdot \frac{x-1}{2}} \left(\frac{d}{c}\right)^m \frac{y}{x} \pmod{p} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Proof. Since $p \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{4}$ we see that $q \nmid x$ and x is odd. We first assume $(c, x+d) = 1$. By Lemma 2.11 we have $(q, (x+d)(c^2 + (x+d)^2)) = 1$. It is easily seen that $\frac{c/(x+d)-i}{c/(x+d)+i} = \frac{c-(x+d)i}{c+(x+d)i} \equiv \frac{c-di}{ix} \pmod{q}$. Thus, for $k = 0, 1, 2, 3$, using Lemma 2.7 we get

$$\begin{aligned} \left(\frac{c+(x+d)i}{q}\right)_4 &= i^k \\ \iff \frac{c}{x+d} \in Q_k(q) &\iff \left(\frac{\frac{c}{x+d}-i}{\frac{c}{x+d}+i}\right)^{\frac{q+1}{4}} \equiv i^k \pmod{q} \\ \iff \left(\frac{c-di}{ix}\right)^{\frac{q+1}{4}} &\equiv i^k \pmod{q} \iff \left(\frac{c-di}{x}\right)^{\frac{q+1}{4}} \equiv i^{\frac{q+1}{4}+k} \pmod{q}. \end{aligned}$$

Since $(\frac{c-di}{x})^{\frac{q+1}{4}} \equiv i^m \pmod{q}$, from the above we deduce that

$$\left(\frac{c+(x+d)i}{q}\right)_4 = i^{m-\frac{q+1}{4}} = \begin{cases} (-1)^{\frac{q+5}{8}} i^{m+1} & \text{if } q \equiv 3 \pmod{8}, \\ (-1)^{\frac{q+1}{8}} i^m & \text{if } q \equiv 7 \pmod{8}. \end{cases}$$

Now, applying Theorem 3.2 we derive the result.

Now we assume $(d_0, x+c) = 1$. By Lemma 2.11, $(q, x+c) = (q, d^2 + (x+c)^2) = 1$. It is easily seen that $\frac{d+(x+c)i}{d-(x+c)i} \equiv \frac{c-di}{-x} \pmod{q}$. Thus, for $k = 0, 1, 2, 3$, using Lemma 2.7 we get

$$\begin{aligned} \left(\frac{d-(x+c)i}{q}\right)_4 &= i^k \\ \iff -\frac{d}{x+c} \in Q_k(q) &\iff \left(\frac{-\frac{d}{x+c}-i}{-\frac{d}{x+c}+i}\right)^{\frac{q+1}{4}} \equiv i^k \pmod{q} \\ \iff \left(\frac{d+(x+c)i}{d-(x+c)i}\right)^{\frac{q+1}{4}} &\equiv i^k \pmod{q} \iff \left(\frac{c-di}{-x}\right)^{\frac{q+1}{4}} \equiv i^k \pmod{q} \\ \iff \left(\frac{c-di}{x}\right)^{\frac{q+1}{4}} &\equiv i^{\frac{q+1}{2}+k} \pmod{q}. \end{aligned}$$

Since $(\frac{c-di}{x})^{\frac{q+1}{4}} \equiv i^m \pmod{q}$, from the above we deduce that $(\frac{d-(x+c)i}{q})_4 = i^{m-\frac{q+1}{2}} = (-1)^{\frac{q+1}{4}} i^m$. Now applying Theorem 3.3 we deduce the result. The proof is now complete.

Corollary 4.1. *Let $p \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{8}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, d + x) = 1$ or $(d_0, x + c) = 1$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{x-1}{2} + \frac{y}{4}} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8} + \frac{x-1}{2} + \frac{y}{4}} \frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8}} \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

Proof. If $x \equiv \pm c \pmod{q}$, then $q \mid d$ and so $(\frac{c-di}{x})^{\frac{q+1}{4}} \equiv (\pm 1)^{\frac{q+1}{4}} = \pm 1 \pmod{q}$. If $x \equiv \pm d \pmod{q}$, then $q \mid c$ and so $(\frac{c-di}{x})^{\frac{q+1}{4}} \equiv (\mp i)^{\frac{q+1}{4}} = \mp(-1)^{\frac{q-3}{8}} i \pmod{q}$. Now applying Theorem 4.1 we deduce the result.

We note that Corollary 4.1 partially settles [S5, Conjecture 4.3].

For example, let p be a prime such that $p \equiv 13 \pmod{24}$ and hence $p = c^2 + d^2 = x^2 + 3y^2$ with $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. If $(c, x + d) = 1$ or $(d_0, x + c) = 1$, then

$$3^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{3}, \\ \mp \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{3}. \end{cases}$$

This partially solves [S4, Conjecture 9.1].

Theorem 4.2. *Let p and q be primes such that $p \equiv 1 \pmod{4}$, $q \equiv 7 \pmod{8}$, $p = c^2 + d^2 = x^2 + qy^2$, $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x + d) = 1$ or $(d_0, x + c) = 1$. Suppose $(\frac{c-di}{c+di})^{\frac{q+1}{8}} \equiv i^m \pmod{q}$. Then*

$$q^{[p/8]} \equiv \begin{cases} (-1)^{\frac{y}{4}} (\frac{d}{c})^m \pmod{p} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{x-1}{2}} (\frac{d}{c})^m \frac{y}{x} \pmod{p} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Proof. Observe that

$$\left(\frac{c-di}{c+di}\right)^{\frac{q+1}{8}} = \frac{(c-di)^{\frac{q+1}{4}}}{(c^2+d^2)^{\frac{q+1}{8}}} = \frac{(c-di)^{\frac{q+1}{4}}}{(x^2+qy^2)^{\frac{q+1}{8}}} \equiv \left(\frac{c-di}{x}\right)^{\frac{q+1}{4}} \pmod{q}.$$

The result follows from Theorem 4.1.

We note that if $q \nmid d$, then the m in Theorem 4.2 depends only on $\frac{c}{d} \pmod{q}$.

Corollary 4.2. *Let $p \equiv 1 \pmod{4}$ and $q \equiv 7 \pmod{8}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd(c^2 - d^2)$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x+d) = 1$ or $(d_0, x+c) = 1$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{q+1}{8} + \frac{y}{4}} \pmod{p} & \text{if } q \mid c, \\ (-1)^{\frac{y}{4}} \pmod{p} & \text{if } q \mid d, \\ \pm(-1)^{\frac{q+9}{16} + \frac{y}{4} \frac{d}{c}} \pmod{p} & \text{if } 16 \mid (q-7) \text{ and } c \equiv \pm d \pmod{q}, \\ (-1)^{\frac{q+1}{16} + \frac{y}{4}} \pmod{p} & \text{if } 16 \mid (q-15) \text{ and } c \equiv \pm d \pmod{q}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{q+1}{8} + \frac{x-1}{2} \frac{y}{x}} \pmod{p} & \text{if } q \mid c, \\ (-1)^{\frac{x-1}{2} \frac{y}{x}} \pmod{p} & \text{if } q \mid d, \\ \pm(-1)^{\frac{q+9}{16} + \frac{x-1}{2} \frac{dy}{cx}} \pmod{p} & \text{if } 16 \mid (q-7) \text{ and } c \equiv \pm d \pmod{q}, \\ (-1)^{\frac{q+1}{16} + \frac{x-1}{2} \frac{y}{x}} \pmod{p} & \text{if } 16 \mid (q-15) \text{ and } c \equiv \pm d \pmod{q}. \end{cases}$$

Proof. Clearly

$$\frac{c-di}{c+di} \equiv \begin{cases} -1 \pmod{q} & \text{if } q \mid c, \\ 1 \pmod{q} & \text{if } q \mid d, \\ -i \pmod{q} & \text{if } c \equiv d \pmod{q}, \\ i \pmod{q} & \text{if } c \equiv -d \pmod{q}. \end{cases}$$

Thus the result follows from Theorem 4.2.

Theorem 4.3. *Let p and q be distinct primes of the form $4k+1$, $p = c^2 + d^2 = x^2 + qy^2$, $q = a^2 + b^2$, $a, b, c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x+d) = 1$ or $(d_0, x+c) = 1$. Suppose $(\frac{ac+bd}{ax})^{\frac{q-1}{4}} \equiv (\frac{b}{a})^m \pmod{q}$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{d}{4} + [\frac{x+2}{4}]} (\frac{d}{c})^m \pmod{p} & \text{if } 2 \mid x, \\ (-1)^{\frac{q-1}{4} \cdot \frac{x-1}{2} + \frac{d}{4} + \frac{y}{4}} (\frac{d}{c})^m \pmod{p} & \text{if } 2 \nmid x. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{[\frac{x}{4}]} (\frac{d}{c})^{m+1} \frac{y}{x} \pmod{p} & \text{if } 2 \mid x, \\ -(-1)^{\frac{q+3}{4} \cdot \frac{x-1}{2}} (\frac{d}{c})^{m+1} \frac{y}{x} \pmod{p} & \text{if } 2 \nmid x. \end{cases}$$

Proof. Clearly $q \nmid x$. We first assume $(c, x + d) = 1$. By Lemma 2.11, $(q, (x + d)(c^2 + (x + d)^2)) = 1$. It is easily seen that $\frac{ac+b(x+d)}{ac-b(x+d)} \equiv \frac{ac+bd}{ax} \cdot \frac{b}{a} \pmod{q}$. Thus, for $k = 0, 1, 2, 3$, using Lemma 2.7 we get

$$\begin{aligned} \left(\frac{c+(x+d)i}{q}\right)_4 = i^k &\iff \frac{c}{x+d} \in Q_k(q) \iff \left(\frac{\frac{c}{x+d} + \frac{b}{a}}{\frac{c}{x+d} - \frac{b}{a}}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \\ &\iff \left(\frac{ac+b(x+d)}{ac-b(x+d)}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \\ &\iff \left(\frac{ac+bd}{ax} \cdot \frac{b}{a}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \\ &\iff \left(\frac{ac+bd}{ax}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^{k-\frac{q-1}{4}} \pmod{q}. \end{aligned}$$

Since $\left(\frac{ac+bd}{ax}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^m \pmod{q}$, from the above we get $\left(\frac{c+(x+d)i}{q}\right)_4 = i^{m+\frac{q-1}{4}}$. Now the result follows from Theorems 3.1 and 3.2 immediately.

Suppose $(d_0, x + c) = 1$. By Lemma 2.11, $(q, (x + c)(d^2 + (x + c)^2)) = 1$. It is easily seen that $\frac{ad-b(x+c)}{ad+b(x+c)} \equiv \frac{ac+bd}{-ax} \pmod{q}$. Thus, for $k = 0, 1, 2, 3$, using Lemma 2.7 we get

$$\begin{aligned} \left(\frac{d-(x+c)i}{q}\right)_4 = i^k &\iff -\frac{d}{x+c} \in Q_k(q) \iff \left(\frac{-\frac{d}{x+c} + \frac{b}{a}}{-\frac{d}{x+c} - \frac{b}{a}}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \\ &\iff \left(\frac{ad-b(x+c)}{ad+b(x+c)}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \iff \left(\frac{ac+bd}{-ax}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^k \pmod{q} \\ &\iff \left(\frac{ac+bd}{ax}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^{\frac{q-1}{2}+k} \pmod{q}. \end{aligned}$$

Since $\left(\frac{ac+bd}{ax}\right)^{\frac{q-1}{4}} \equiv \left(\frac{b}{a}\right)^m \pmod{q}$, by the above we get $\left(\frac{d-(x+c)i}{q}\right)_4 = i^{m-\frac{q-1}{2}}$. Thus, applying Theorem 3.3 and the fact $\frac{x}{2} \equiv \frac{x^2}{4} = \frac{c^2-xy^2+d^2}{4} \equiv \frac{1-q}{4} + \frac{d}{2} \pmod{2}$ for even x we derive the result. The proof is now complete.

Corollary 4.3. *Let $p \equiv 1 \pmod{4}$ and $q \equiv 5 \pmod{8}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x + d) = 1$ or $(d_0, x + c) = 1$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$q^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{d}{4} + \frac{x+2}{4}} \pmod{p} & \text{if } 2 \mid x \text{ and } x \equiv \pm c \pmod{q}, \\ \pm(-1)^{\frac{d}{4} + \frac{x-1}{2} + \frac{y}{4}} \pmod{p} & \text{if } 2 \nmid x \text{ and } x \equiv \pm c \pmod{q}, \\ \pm(-1)^{\frac{q-5}{8} + \frac{d}{4} + \frac{x+2}{4} \frac{d}{c}} \pmod{p} & \text{if } 2 \mid x \text{ and } x \equiv \pm d \pmod{q}, \\ \pm(-1)^{\frac{q-5}{8} + \frac{d}{4} + \frac{x-1}{2} + \frac{y}{4} \frac{d}{c}} \pmod{p} & \text{if } 2 \nmid x \text{ and } x \equiv \pm d \pmod{q}. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \pm \delta(x) \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp (-1)^{\frac{q-5}{8}} \delta(x) \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}, \end{cases}$$

where $\delta(x) = 1$ or -1 according as $8 \mid x$ or not.

Proof. If $x \equiv \pm c \pmod{q}$, then $q \mid d$ and so $(\frac{ac+bd}{ax})^{\frac{q-1}{4}} \equiv (\frac{c}{x})^{\frac{q-1}{4}} \equiv (\pm 1)^{\frac{q-1}{4}} = \pm 1 \pmod{q}$. If $x \equiv \pm d \pmod{q}$, then $q \mid c$ and so $(\frac{ac+bd}{ax})^{\frac{q-1}{4}} \equiv (\frac{bd}{ax})^{\frac{q-1}{4}} \equiv (\pm \frac{b}{a})^{\frac{q-1}{4}} \equiv \pm (-1)^{\frac{q-5}{8}} \frac{b}{a} \pmod{q}$. Now putting the above with Theorem 4.3 we deduce the result.

Theorem 4.4. Let p and q be distinct primes such that $p \equiv 1 \pmod{4}$, $q \equiv 1 \pmod{8}$, $p = c^2 + d^2 = x^2 + qy^2$, $q = a^2 + b^2$, $a, b, c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x+d) = 1$ or $(d_0, x+c) = 1$. Suppose $(\frac{ac+bd}{ac-bd})^{\frac{q-1}{8}} \equiv (\frac{b}{a})^m \pmod{q}$.

(i) If $p \equiv 1 \pmod{8}$, then

$$q^{\frac{p-1}{8}} \equiv (-1)^{\frac{d}{4} + \frac{xy}{4}} \left(\frac{d}{c}\right)^m \pmod{p}.$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{x-2}{4}} \left(\frac{d}{c}\right)^{m+1} \frac{y}{x} \pmod{p} & \text{if } 2 \mid x, \\ (-1)^{\frac{x+1}{2}} \left(\frac{d}{c}\right)^{m+1} \frac{y}{x} \pmod{p} & \text{if } 2 \nmid x. \end{cases}$$

Proof. Observe that $b^2 \equiv -a^2 \pmod{q}$, $p \equiv x^2 \pmod{q}$ and so

$$\left(\frac{ac+bd}{ac-bd}\right)^{\frac{q-1}{8}} = \frac{(ac+bd)^{\frac{q-1}{4}}}{(a^2c^2 - b^2d^2)^{\frac{q-1}{8}}} \equiv \frac{(ac+bd)^{\frac{q-1}{4}}}{(a^2p)^{\frac{q-1}{8}}} \equiv \left(\frac{ac+bd}{ax}\right)^{\frac{q-1}{4}} \pmod{q}.$$

The result follows from Theorem 4.3.

We note that if $q \nmid d$, then the m in Theorem 4.4 depends only on $\frac{c}{d} \pmod{q}$.

Corollary 4.4. Let $p \equiv 1 \pmod{4}$ and $q \equiv 1 \pmod{8}$ be distinct primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd(c^2 - d^2)$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x+d) = 1$ or $(d_0, x+c) = 1$.

(i) If $p \equiv 1 \pmod{8}$, then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8} + \frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } q \mid c, \\ (-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } q \mid d, \\ (-1)^{\frac{q-1}{16} + \frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 16 \mid (q-1) \text{ and } c \equiv \pm d \pmod{q}, \\ \pm (-1)^{\frac{q-9}{16} + \frac{d}{4} + \frac{xy}{4}} \frac{d}{c} \pmod{p} & \text{if } 16 \mid (q-9) \text{ and } c \equiv \pm d \pmod{q}. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$ and $\varepsilon(x) = (-1)^{\frac{x-2}{4}}$ or $(-1)^{\frac{x+1}{2}}$ according as $2 \mid x$ or $2 \nmid x$, then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8}} \varepsilon(x) \frac{dy}{cx} \pmod{p} & \text{if } q \mid c, \\ \varepsilon(x) \frac{dy}{cx} \pmod{p} & \text{if } q \mid d, \\ (-1)^{\frac{q-1}{16}} \varepsilon(x) \frac{dy}{cx} \pmod{p} & \text{if } 16 \mid (q-1) \text{ and } c \equiv \pm d \pmod{q}, \\ \mp (-1)^{\frac{q-9}{16}} \varepsilon(x) \frac{y}{x} \pmod{p} & \text{if } 16 \mid (q-9) \text{ and } c \equiv \pm d \pmod{q}. \end{cases}$$

Proof. Suppose that $q = a^2 + b^2$ with $a, b \in \mathbb{Z}$. Then clearly

$$\left(\frac{ac + bd}{ac - bd} \right)^{\frac{q-1}{8}} \equiv \begin{cases} (-1)^{\frac{q-1}{8}} \pmod{q} & \text{if } q \mid c, \\ 1 \pmod{q} & \text{if } q \mid d, \\ (-1)^{\frac{q-1}{16}} \pmod{q} & \text{if } 16 \mid (q-1) \text{ and } c \equiv \pm d \pmod{q}, \\ \pm (-1)^{\frac{q-9}{16}} \frac{b}{a} \pmod{q} & \text{if } 16 \mid (q-9) \text{ and } c \equiv \pm d \pmod{q}. \end{cases}$$

Thus the result follows from Theorem 4.4.

Corollary 4.5. Let $p \equiv 1 \pmod{4}$ be a prime such that $p \neq 17$ and $p = c^2 + d^2 = x^2 + 17y^2$ with $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x + d) = 1$ or $(d_0, x + c) = 1$.

(i) If $p \equiv 1 \pmod{8}$, then

$$17^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 17 \mid cd, \\ -(-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } c \equiv \pm d \pmod{17}, \\ \pm (-1)^{\frac{d}{4} + \frac{xy}{4}} \frac{c}{d} \pmod{p} & \text{if } c \equiv \pm 5d, \pm 10d \pmod{17}. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$ and $\varepsilon(x) = (-1)^{\frac{x-2}{4}}$ or $(-1)^{\frac{x+1}{2}}$ according as $2 \mid x$ or $2 \nmid x$, then

$$17^{\frac{p-5}{8}} \equiv \begin{cases} \varepsilon(x) \frac{dy}{cx} \pmod{p} & \text{if } 17 \mid cd, \\ -\varepsilon(x) \frac{dy}{cx} \pmod{p} & \text{if } c \equiv \pm d \pmod{17}, \\ \pm \varepsilon(x) \frac{y}{x} \pmod{p} & \text{if } c \equiv \pm 5d, \pm 10d \pmod{17}. \end{cases}$$

Proof. Since $17 = 1^2 + 4^2$ and $\left(\frac{\pm 5+4}{\pm 5-4}\right)^{\frac{17-1}{8}} \equiv \left(\frac{\pm 10+4}{\pm 10-4}\right)^{\frac{17-1}{8}} \equiv \mp 4 \pmod{17}$, from Theorem 4.4 and Corollary 4.4 we deduce the result.

Theorem 4.5. Let $p \equiv 1 \pmod{4}$ be a prime, $p = c^2 + d^2 = x^2 + (a^2 + b^2)y^2 \neq a^2 + b^2$, $a, b, c, d, x, y \in \mathbb{Z}$, $a \neq 0$, $2 \mid a$, $(a, b) = 1$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$, $y = 2^t y_0$ and $d_0 \equiv y_0 \equiv 1 \pmod{4}$. Assume $(c, x + d) = 1$ or $(d_0, x + c) = 1$. Suppose $\left(\frac{ac+bd}{b+ai}\right)_4 = i^m$.

(i) If $p \equiv 1 \pmod{8}$, then

$$(a^2 + b^2)^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{d}{4} + \frac{x}{4}} \left(\frac{c}{d}\right)^m \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \mid x, \\ (-1)^{\frac{d}{4} + \frac{y}{4}} \left(\frac{c}{d}\right)^m \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \nmid x, \\ (-1)^{\frac{b+1}{2} + \frac{d}{4} + \frac{x-2}{4}} \left(\frac{c}{d}\right)^{m-1} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \mid x, \\ (-1)^{\frac{b-1}{2} + \frac{d}{4} + \frac{y}{4} + \frac{x-1}{2}} \left(\frac{c}{d}\right)^{m-1} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \nmid x. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$(a^2 + b^2)^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^{\frac{x-2}{4}} \left(\frac{c}{d}\right)^{m-1} \frac{y}{x} \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \mid x, \\ (-1)^{\frac{x+1}{2}} \left(\frac{c}{d}\right)^{m-1} \frac{y}{x} \pmod{p} & \text{if } 4 \mid a \text{ and } 2 \nmid x, \\ (-1)^{\frac{x}{4} + \frac{b+1}{2}} \left(\frac{c}{d}\right)^m \frac{y}{x} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \mid x, \\ (-1)^{\frac{b-1}{2}} \left(\frac{c}{d}\right)^m \frac{y}{x} \pmod{p} & \text{if } 2 \parallel a \text{ and } 2 \nmid x. \end{cases}$$

Proof. Set $q = a^2 + b^2$. Then clearly $2 \nmid q$ and $p \nmid q$. We first assume $(c, x+d) = 1$. By Lemma 2.11, $(q, x+d) = (q, c^2 + (x+d)^2) = 1$. Since $\frac{c-(x+d)i}{c+(x+d)i} \equiv \frac{c-di}{ix} \pmod{q}$, we see that

$$\begin{aligned} & \left(\frac{c/(x+d) + i}{q}\right)_4 \\ &= \left(\frac{c + (x+d)i}{q}\right)_4 = \left(\frac{c + (x+d)i}{b + ai}\right)_4 \left(\frac{c + (x+d)i}{b - ai}\right)_4 \\ &= \left(\frac{c + (x+d)i}{b + ai}\right)_4 \overline{\left(\frac{c - (x+d)i}{b + ai}\right)_4} = \left(\frac{c + (x+d)i}{b + ai}\right)_4 \left(\frac{c - (x+d)i}{b + ai}\right)_4^{-1} \\ &= \left(\frac{c-(x+d)i}{c+(x+d)i}\right)_4^{-1} = \left(\frac{c-di}{ix}\right)_4^{-1} = \left(\frac{ai}{b+ai}\right)_4 \left(\frac{(ac-adi)/x}{b+ai}\right)_4^{-1} \\ &= \left(\frac{-b}{b+ai}\right)_4 \left(\frac{(ac+bd)/x}{b+ai}\right)_4^{-1} = (-1)^{\frac{b+1}{2} \cdot \frac{a}{2}} \left(\frac{b+ai}{b}\right)_4 i^{-m} \\ &= (-1)^{\frac{b+1}{2} \cdot \frac{a}{2}} \left(\frac{i}{b}\right)_4 i^{-m} = (-1)^{\frac{b+1}{2} \cdot \frac{a}{2} + \frac{b^2-1}{8}} i^{-m} \\ &= (-1)^{\frac{b+1}{2} \cdot \frac{a}{2} + \frac{a-1-a^2}{8}} i^{-m} = (-1)^{\frac{b+1}{2} \cdot \frac{a}{2} + \lfloor \frac{a}{8} \rfloor} i^{-m}. \end{aligned}$$

This together with Theorems 3.1 and 3.2 yields the result under the condition $(c, x+d) = 1$.

Now we assume $(d_0, x+c) = 1$. By Lemma 2.11, $(q, x+c) = (q, (x+c)^2 + d^2) = 1$.

Since $\frac{d+(x+c)i}{d-(x+c)i} \equiv \frac{c-di}{-x} \pmod{q}$, using Lemma 2.6 we see that

$$\begin{aligned}
& \left(\frac{d/(x+c) - i}{q} \right)_4 \\
&= \left(\frac{d - (x+c)i}{q} \right)_4 = \left(\frac{d - (x+c)i}{b+ai} \right)_4 \left(\frac{d - (x+c)i}{b-ai} \right)_4 \\
&= \left(\frac{d - (x+c)i}{b+ai} \right)_4 \overline{\left(\frac{d + (x+c)i}{b+ai} \right)_4} = \left(\frac{d - (x+c)i}{b+ai} \right)_4 \left(\frac{d + (x+c)i}{b+ai} \right)_4^{-1} \\
&= \left(\frac{d+(x+c)i}{d-(x+c)i} \right)_4^{-1} = \left(\frac{c-di}{-x} \right)_4^{-1} = \left(\frac{-a}{b+ai} \right)_4 \left(\frac{(ac-adi)/x}{b+ai} \right)_4^{-1} \\
&= (-1)^{\frac{a}{2}} \left(\frac{a}{b+ai} \right)_4 \left(\frac{(ac+bd)/x}{b+ai} \right)_4^{-1} = (-1)^{\frac{a}{2}} \left(\frac{a}{b+ai} \right)_4 i^{-m} \\
&= \begin{cases} -(-1)^{\frac{b-1}{2}} i \cdot i^{-m} = (-1)^{\frac{b+1}{2}} i^{1-m} & \text{if } 2 \parallel a, \\ 1 \cdot 1 \cdot i^{-m} = i^{-m} & \text{if } 4 \mid a. \end{cases}
\end{aligned}$$

Combining this with Theorem 3.3 we deduce the result under the condition $(d_0, x+c) = 1$. The proof is now complete.

Remark 4.1 Let p be a prime of the form $4k+1$, $q \in \mathbb{Z}$, $2 \nmid q$, $p \nmid q$, and $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$, $c \equiv 1 \pmod{4}$, $d = 2^r d_0$ and $d_0 \equiv 1 \pmod{4}$, we conjecture that one can always choose the sign of x such that $(c, x+d) = 1$ or $(d_0, x+c) = 1$. Thus the condition $(c, x+d) = 1$ or $(d_0, x+c) = 1$ in Theorems 4.1-4.5 and Corollaries 4.1-4.5 can be canceled. See also related conjectures in [S4] and [S5].

REFERENCES

- [BEW] B.C. Berndt, R.J. Evans and K.S. Williams, *Gauss and Jacobi Sums*, Wiley, New York, 1998.
- [E] R.J. Evans, *Residuacity of primes*, Rocky Mountain J. Math. **19** (1989), 1069-1081.
- [H] R.H. Hudson, *Diophantine determinations of $3^{(p-1)/8}$ and $5^{(p-1)/4}$* , Pacific J. Math. **111** (1984), 49-55.
- [HW] R.H. Hudson and K.S. Williams, *Some new residuacity criteria*, Pacific J. Math. **91** (1980), 135-143.
- [IR] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer, New York, 1990.
- [Lem] F. Lemmermeyer, *Reciprocity Laws: From Euler to Eisenstein*, Springer, Berlin, 2000.
- [S1] Z.H. Sun, *Supplements to the theory of quartic residues*, Acta Arith. **97** (2001), 361-377.
- [S2] Z.H. Sun, *Quartic residues and binary quadratic forms*, J. Number Theory **113** (2005), 10-52.
- [S3] Z.H. Sun, *On the quadratic character of quadratic units*, J. Number Theory **128** (2008), 1295-1335.
- [S4] Z.H. Sun, *Quartic, octic residues and Lucas sequences*, J. Number Theory **129** (2009), 499-550.
- [S5] Z.H. Sun, *Congruences for $(A + \sqrt{A^2 + mB^2})^{(p-1)/2}$ and $(b + \sqrt{a^2 + b^2})^{(p-1)/4} \pmod{p}$* , Acta Arith. **149** (2011), 275-296.