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On the number of representations of n as a linear combination of four triangular numbers

Min Wang
School of Mathematical Sciences
Suzhou, Jiangsu 215006, P.R. China
20144207002@stu.suda.edu.cn

Zhi-Hong Sun*
School of Mathematical Sciences
Huaiyin Normal University
Huaian, Jiangsu 223001, P.R. China
zhsun@hytc.edu.cn

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Abstract

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. For $a,b,c,d,n\in\mathbb{N}$ let $t(a,b,c,d;n)$ be the number of representations of n by $ax(x-1)/2+by(y-1)/2+cz(z-1)/2+dw(w-1)/2$ $(x,y,z,w\in\mathbb{Z})$. In this paper we obtain explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d)=(1,2,2,4)$, $(1,2,4,4)$, $(1,1,4,4)$, $(1,4,4,4)$, $(1,3,9,9)$, $(1,1,3,9)$, $(1,3,3,9)$, $(1,1,9,9)$, $(1,9,9,9)$ and $(1,1,1,9)$.

Keywords: Representation; triangular number

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1. Introduction

Let $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^4=\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{N}^4=\mathbb{N}\times\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. For $n\in\mathbb{N}$ let

$$\sigma(n)=\sum_{d\mid n,d\in\mathbb{N}} d.$$

For convenience we define $\sigma(n)=0$ for $n\notin\mathbb{N}$. For $a,b,c,d\in\mathbb{N}$ and $n\in\mathbb{N}\cup\{0\}$ set

$$N(a,b,c,d;n)=\left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n=ax^2+by^2+cz^2+dw^2\}\right|$$

*Corresponding author.

and

$$t(a,b,c,d;n)=\Big|\Big\{(x,y,z,w)\in\mathbb{Z}^4 \mid n\ =a\frac{x(x-1)}{2}+b\frac{y(y-1)}{2}+c\frac{z(z-1)}{2}+d\frac{w(w-1)}{2}\Big\}\Big|.$$

The numbers $\frac{x(x-1)}{2}$ $(x\in\mathbb{Z})$ are called triangular numbers.

In 1828 Jacobi showed that

$$N(1,1,1,1;n)=8\sum_{d\mid n,4\nmid d} d. \tag{1.1}$$

In 1847 Eisenstein (see [11]) gave formulas for $N(1,1,1,3;n)$ and $N(1,1,1,5;n)$. From 1859 to 1866 Liouville made about 90 conjectures on $N(a,b,c,d;n)$ in a series of papers. Most conjectures of Liouville have been proved. See [2-7], Cooper's survey paper [10], Dickson's historical comments [11] and Williams' book [16].

Let

$$t'(a,b,c,d;n)=\Big|\Big\{(x,y,z,w)\in\mathbb{N}^4 \mid n=a\frac{x(x-1)}{2}+b\frac{y(y-1)}{2}+c\frac{z(z-1)}{2}+d\frac{w(w-1)}{2}\Big\}\Big|.$$

As $\frac{1}{2}x(x-1)=\frac{1}{2}(-x+1)(-x)$ we have

$$t(a,b,c,d;n)=16t'(a,b,c,d;n).$$

In [13] Legendre stated that

$$t'(1,1,1,1;n)=\sigma(2n+1). \tag{1.2}$$

In 2003, Williams [15] showed that

$$t'(1,1,2,2;n)=\frac{1}{4}\sum_{d\mid 4n+3}\big(d-(-1)^{\frac{d-1}{2}}\big).$$

For $a,b,c,d\in\mathbb{N}$ with $4<a+b+c+d\le 8$ let

$$C(a,b,c,d)=16+4i_1(i_1-1)i_2+8i_1i_3,$$

where i_j is the number of elements in $\{a,b,c,d\}$ which are equal to j. When $4<a+b+c+d\le 7$, in 2005 Adiga, Cooper and Han [1] showed that

$$C(a,b,c,d)t'(a,b,c,d;n)=N(a,b,c,d;8n+a+b+c+d). \tag{1.3}$$

When $a+b+c+d=8$, in 2008 Baruah, Cooper and Hirschhorn [8] proved that

$$C(a,b,c,d)t'(a,b,c,d;n)=N(a,b,c,d;8n+8)-N(a,b,c,d;2n+2). \tag{1.4}$$

In 2009, Cooper [10] determined $t'(a,b,c,d;n)$ for $(a,b,c,d)=(1,1,1,3)$, $(1,3,3,3)$, $(1,2,2,3)$, $(1,3,6,6)$, $(1,3,4,4)$, $(1,1,2,6)$ and $(1,3,12,12)$.

In this paper, by using some formulas for $N(a,b,c,d;n)$ in [2-7] and Ramanujan's theta functions we obtain explicit formulas for $t(a,b,c,d;n)$ in the cases $(a,b,c,d)=(1,2,2,4)$, $(1,2,4,4)$, $(1,1,4,4)$, $(1,4,4,4)$, $(1,3,3,9)$, $(1,1,9,9)$, $(1,9,9,9)$, $(1,1,1,9)$, $(1,3,9,9)$ and $(1,1,3,9)$.

For $m, n \in \mathbb{N}$ with $2 \mid m$ and $2 \nmid n$ we define

$$S_m(n) = \sum_{\substack{(r,s)\in\mathbb{Z}\times\mathbb{Z} \\ n=r^2+ms^2 \\ r\equiv 1 \pmod 4}} r.$$

As $r^2 + 2s^2 \equiv 0, 1, 2, 3, 4, 6 \pmod 8$ for $r, s \in \mathbb{Z}$, we see that $S_2(n) = 0 \quad \text{for} \quad n \equiv 5, 7 \pmod 8$. Also, $r^2 + 4s^2 \equiv 0, 1 \pmod 4$ for $r, s \in \mathbb{Z}$ implies that $S_4(n) = 0 \quad \text{for} \quad n \equiv 2, 3 \pmod 4$. In this paper, following [7] we also define

$$S(n) = \sum_{d\mid n} \frac{n}{d}\Big(\frac{2}{d}\Big),$$

where $(\frac{a}{m})$ is the Legendre-Jacobi-Kronecker symbol.

2. Formulas for $t(1, 3, 9, 9; n)$ and $t(1, 1, 3, 9; n)$

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \ (|q| < 1).$$

It is evident that for $|q| < 1$,

$$\sum_{n=0}^{\infty} N(a, b, c, d; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d),$$

$$\sum_{n=0}^{\infty} t'(a, b, c, d; n)q^n = \psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

From [8, Lemma 4.1] we know that for $|q| < 1$,

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8) \tag{2.1}$$

and

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\psi(q^{12})\varphi(q^2). \tag{2.2}$$

Theorem 2.1. *Let $n \in \mathbb{N}$. Then*

$$N(1, 3, 9, 9; 8n + 22) = 40t'(1, 3, 9, 9; n).$$

Proof. By (2.1), for $|q| < 1$ we have

$$\varphi(q^k) = \varphi(q^{4k}) + 2q^k\psi(q^{8k}) = \varphi(q^{16k}) + 2q^{4k}\psi(q^{32k}) + 2q^k\psi(q^{8k}).$$

Thus, for $|q| < 1$ we have

$$\sum_{n=0}^{\infty} N(1, 3, 9, 9; n)q^n$$

$$
\begin{aligned}
&= \varphi(q)\varphi(q^3)\varphi(q^9)^2\\
&= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24}))\\
&\quad \times (\varphi(q^{144}) + 2q^{36}\psi(q^{288}) + 2q^9\psi(q^{72}))^2\\
&= \big(\varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\psi(q^8)\varphi(q^{48}) + 2q^3\psi(q^{24})\varphi(q^{16})\\
&\quad + 2q^4\psi(q^{32})\varphi(q^{48}) + 4q^4\psi(q^8)\psi(q^{24}) + 2q^{12}\psi(q^{96})\varphi(q^{16}) + 4q^7\psi(q^{24})\psi(q^{32})\\
&\quad + 4q^{13}\psi(q^8)\psi(q^{96})\big)\big(\varphi(q^{144})^2 + 4q^{72}\psi(q^{288})^2 + 4q^{36}\varphi(q^{144})\psi(q^{288}) + 4q^{18}\psi(q^{72})^2\\
&\quad + 4q^9\varphi(q^{144})\psi(q^{72}) + 8q^{45}\psi(q^{288})\psi(q^{72})\big).
\end{aligned}
$$

Since

$$
\varphi(q^{8k}) = 1 + 2\sum_{n=1}^{\infty} q^{8kn^2} \quad \text{and} \quad \psi(q^{8k}) = \sum_{n=0}^{\infty} q^{8kn(n+1)/2} \ (|q| < 1),
$$

we see that for any nonnegative integers k_1, k_2, m_1 and m_2,

$$
\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2} = \sum_{n=0}^{\infty} b_n q^{8n} \quad (|q| < 1).
$$

Now from the above we deduce that for $|q| < 1$,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} N(1,3,9,9;8n+6)q^{8n+6}\\
&= 2q\psi(q^8)\varphi(q^{48}) \cdot 8q^{45}\psi(q^{288})\psi(q^{72}) + 2q^4\psi(q^{32})\varphi(q^{48}) \cdot 4q^{18}\psi(q^{72})^2\\
&\quad + 4q^4\psi(q^8)\psi(q^{24}) \cdot 4q^{18}\psi(q^{72})^2 + 2q^{12}\psi(q^{96})\varphi(q^{16}) \cdot 4q^{18}\psi(q^{72})^2\\
&\quad + 4q^{13}\psi(q^8)\psi(q^{96}) \cdot 4q^9\varphi(q^{144})\psi(q^{72})\\
&= 16q^{46}\varphi(q^{48})\psi(q^8)\psi(q^{72})\psi(q^{288}) + 8q^{22}\varphi(q^{48})\psi(q^{32})\psi(q^{72})^2 + 16q^{22}\psi(q^8)\psi(q^{24})\psi(q^{72})^2\\
&\quad + 8q^{30}\varphi(q^{16})\psi(q^{96})\psi(q^{72})^2 + 16q^{22}\varphi(q^{144})\psi(q^8)\psi(q^{72})\psi(q^{96})
\end{aligned}
$$

and so

$$
\begin{aligned}
&\frac{1}{8}\sum_{n=0}^{\infty} N(1,3,9,9;8n+6)q^{8n-16}\\
&= 2q^{24}\varphi(q^{48})\psi(q^8)\psi(q^{72})\psi(q^{288}) + \varphi(q^{48})\psi(q^{32})\psi(q^{72})^2 + 2\psi(q^8)\psi(q^{24})\psi(q^{72})^2\\
&\quad + q^8\varphi(q^{16})\psi(q^{96})\psi(q^{72})^2 + 2\varphi(q^{144})\psi(q^8)\psi(q^{72})\psi(q^{96}).
\end{aligned}
$$

Replacing q with $q^{1/8}$ in the above we obtain

$$
\begin{aligned}
&\frac{1}{8}\sum_{n=0}^{\infty} N(1,3,9,9;8n+22)q^n\\
&= \frac{1}{8}\sum_{n=0}^{\infty} N(1,3,9,9;8n+6)q^{n-2}\\
&= 2q^3\varphi(q^6)\psi(q)\psi(q^9)\psi(q^{36}) + \varphi(q^6)\psi(q^4)\psi(q^9)^2 + 2\psi(q)\psi(q^3)\psi(q^9)^2\\
&\quad + q\varphi(q^2)\psi(q^{12})\psi(q^9)^2 + 2\varphi(q^{18})\psi(q)\psi(q^9)\psi(q^{12}).
\end{aligned}
$$

Now applying (2.2) we get

$$\begin{aligned}&\frac{1}{8}\sum_{n=0}^{\infty}N(1,3,9,9;8n+22)q^n\\&=2\psi(q)\psi(q^3)\psi(q^9)^2+\psi(q^9)^2\psi(q)\psi(q^3)+2\psi(q)\psi(q^9)\psi(q^3)\psi(q^9)\\&=5\psi(q)\psi(q^3)\psi(q^9)^2=5\sum_{n=0}^{\infty}t'(1,3,9,9;n)q^n.\end{aligned}$$

Comparing the coefficients of q^n in the above expansion we obtain the result. □

For $n\in\mathbb{N}$ following [6] we define

$$\begin{aligned}&A(n)=\sum_{d\mid n}d\Big(\frac{12}{n/d}\Big),\quad B(n)=\sum_{d\mid n}d\Big(\frac{-3}{d}\Big)\Big(\frac{-4}{n/d}\Big),\\&C(n)=\sum_{d\mid n}d\Big(\frac{-3}{n/d}\Big)\Big(\frac{-4}{d}\Big)\quad\text{and}\quad D(n)=\sum_{d\mid n}d\Big(\frac{12}{d}\Big).\end{aligned}$$

Let (a,b) be the greatest common divisor of integers a and b. Suppose that $n\in\mathbb{N}$ and $n=2^\alpha 3^\beta n_1$, where α and β are non-negative integers, $n_1\in\mathbb{N}$ and $(n_1,6)=1$. From [6, Theorem 3.1] we know that

$$\begin{aligned}&A(n)=2^\alpha 3^\beta A(n_1),\quad B(n)=(-1)^{\alpha+\beta}2^\alpha\Big(\frac{-3}{n_1}\Big)A(n_1),\\&C(n)=(-1)^{\alpha+\beta+\frac{n_1-1}{2}}3^\beta A(n_1)\quad\text{and}\quad D(n)=\Big(\frac{3}{n_1}\Big)A(n_1).\end{aligned}\tag{2.3}$$

Lemma 2.1 ([2, Theorem 1.2]). *Let $n\in\mathbb{N}$. Then*

$$N(1,3,9,9;n)=\begin{cases}2A(n/3)+2B(n/3)-C(n/3)-D(n/3) & \text{if } n\equiv 0 \pmod 3,\\ 2A(n)-\frac{2}{3}B(n)+C(n)-\frac{1}{3}D(n) & \text{if } n\equiv 1 \pmod 3,\\ 0 & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

Theorem 2.2. *Let $n\in\mathbb{N}$. Then*

$$t(1,3,9,9;n)=\begin{cases}0 & \text{if } 3\mid n-2,\\ \frac{4}{3}\sum_{d\mid 4n+11}d(\frac{3}{d}) & \text{if } 3\mid n,\\ 2\big(3^{\beta-1}(\frac{3}{n_1})-1\big)\sum_{d\mid n_1}d(\frac{3}{d}) & \text{if } 3\mid n-1 \text{ and } 4n+11=3^\beta n_1\ (3\nmid n_1).\end{cases}$$

Proof. By Theorem 2.1,

$$t(1,3,9,9;n)=16t'(1,3,9,9;n)=\frac{2}{5}N(1,3,9,9;8n+22).$$

Now applying Lemma 2.1 and (2.3) we deduce that

$$t(1,3,9,9;n)=\begin{cases}0 & \text{if } 3\mid n-2,\\ \frac{4}{3}A(4n+11) & \text{if } 3\mid n,\\ 2(3^{\beta-1}-(\frac{3}{n_1}))A(n_1) & \text{if } 3\mid n-1 \text{ and } 4n+11=3^{\beta}n_1\ (3\nmid n_1).\end{cases}$$

To see the result, we note that

$$A(m)=\sum_{d\mid m}d\Big(\frac{12}{m}\Big)\Big(\frac{12}{d}\Big)=\Big(\frac{3}{m}\Big)\sum_{d\mid m}d\Big(\frac{3}{d}\Big)\quad \text{for } m\in\mathbb{N} \text{ with } (6,m)=1. \quad \square \tag{2.4}$$

Lemma 2.2 ([2, Theorem 1.3]). *Let $n\in\mathbb{N}$. Then*

$$N(1,1,3,9;n)=\begin{cases}2A(n/3)+2B(n/3)-C(n/3)-D(n/3) & \text{if } n\equiv 0 \pmod 3,\\ 4A(n)-\frac{4}{3}B(n)+2C(n)-\frac{2}{3}D(n) & \text{if } n\equiv 1 \pmod 3,\\ 2A(n)-\frac{2}{3}B(n)+C(n)-\frac{1}{3}D(n) & \text{if } n\equiv 2 \pmod 3.\end{cases}$$

Theorem 2.3. *Let $n\in\mathbb{N}$. Then*

$$t(1,1,3,9;n)=\begin{cases}-\frac{8}{3}\sum_{d\mid 4n+7}d(\frac{3}{d}) & \text{if } 3\mid n,\\ \frac{8}{3}\sum_{d\mid 4n+7}d(\frac{3}{d}) & \text{if } 3\mid n-1,\\ 2\big(3^{\beta-1}(\frac{3}{n_1})-1\big)\sum_{d\mid n_1}d(\frac{3}{d}) & \text{if } 3\mid n-2 \text{ and } 4n+7=3^{\beta}n_1\ (3\nmid n_1).\end{cases}$$

Proof. Suppose $|q|<1$. Then clearly

$$\sum_{n=0}^{\infty}N(1,1,3,9;n)q^n=\varphi(q)^2\varphi(q^3)\varphi(q^9).$$

Since $\varphi(q^k)=\varphi(q^{4k})+2q^k\psi(q^{8k})=\varphi(q^{16k})+2q^{4k}\psi(q^{32k})+2q^k\psi(q^{8k})$ by (2.1), we see that

$$\begin{aligned}&\varphi(q)^2\varphi(q^3)\varphi(q^9)\\ &=\big(\varphi(q^{16})+2q^4\psi(q^{32})+2q\psi(q^8)\big)^2\big(\varphi(q^{48})+2q^{12}\psi(q^{96})+2q^3\psi(q^{24})\big)\\ &\quad\times\big(\varphi(q^{144})+2q^{36}\psi(q^{288})+2q^9\psi(q^{72})\big)\\ &=\big(\varphi(q^{16})^2+4q^4\psi(q^{32})\varphi(q^{16})+4q^8\psi(q^{32})^2\\ &\quad+4q^2\psi(q^8)^2+4q\varphi(q^{16})\psi(q^8)+8q^5\psi(q^8)\psi(q^{32})\big)\\ &\quad\times\big(\varphi(q^{48})\varphi(q^{144})+2q^{36}\varphi(q^{48})\psi(q^{288})+2q^9\varphi(q^{48})\psi(q^{72})+2q^{12}\psi(q^{96})\varphi(q^{144})\\ &\quad+4q^{48}\psi(q^{96})\psi(q^{288})+4q^{21}\psi(q^{96})\psi(q^{72})+2q^3\psi(q^{24})\varphi(q^{144})\\ &\quad+4q^{39}\psi(q^{24})\psi(q^{288})+4q^{12}\psi(q^{24})\psi(q^{72})\big).\end{aligned}$$

Note that $\varphi(q^{8k_1})^{m_1}\psi(q^{8k_2})^{m_2} = \sum_{n=0}^{\infty} b_n q^{8n}$ for $|q| < 1$ and any nonnegative integers k_1, k_2, m_1 and m_2. From the above we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} N(1,1,3,9;8n+6)q^{8n+6} \\
&= 4q\varphi(q^{16})\psi(q^8)\cdot 4q^{21}\psi(q^{96})\psi(q^{72}) + 8q^5\psi(q^8)\psi(q^{32})\cdot 2q^9\varphi(q^{48})\psi(q^{72}) \\
&\quad + 4q^2\psi(q^8)^2\cdot 2q^{36}\varphi(q^{48})\psi(q^{288}) + 4q^2\psi(q^8)^2\cdot 2q^{12}\psi(q^{96})\varphi(q^{144}) \\
&\quad + 4q^2\psi(q^8)^2\cdot 4q^{12}\psi(q^{24})\psi(q^{72})
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{8}\sum_{n=0}^{\infty} N(1,1,3,9;8n+6)q^{8n-8} \\
&= 2q^8\varphi(q^{16})\psi(q^8)\psi(q^{72})\psi(q^{96}) + 2\psi(q^8)\psi(q^{32})\varphi(q^{48})\psi(q^{72}) \\
&\quad + q^{24}\psi(q^8)^2\varphi(q^{48})\psi(q^{288}) + \psi(q^8)^2\psi(q^{96})\varphi(q^{144}) + 2\psi(q^8)^2\psi(q^{24})\psi(q^{72}).
\end{aligned}$$

Replacing q with $q^{1/8}$ in the above we obtain

$$\begin{aligned}
&\frac{1}{8}\sum_{n=0}^{\infty} N(1,1,3,9;8n+14)q^n \\
&= \frac{1}{8}\sum_{n=0}^{\infty} N(1,1,3,9;8n+6)q^{n-1} \\
&= 2q\varphi(q^2)\psi(q)\psi(q^9)\psi(q^{12}) + 2\psi(q)\psi(q^4)\varphi(q^6)\psi(q^9) + q^3\psi(q)^2\varphi(q^6)\psi(q^{36}) \\
&\quad + \psi(q)^2\psi(q^{12})\varphi(q^{18}) + 2\psi(q)^2\psi(q^3)\psi(q^9).
\end{aligned}$$

Now applying (2.2) we get

$$\begin{aligned}
&\frac{1}{8}\sum_{n=0}^{\infty} N(1,1,3,9;8n+14)q^n \\
&= 2\psi(q)^2\psi(q^3)\psi(q^9) + \psi(q)^2\psi(q^3)\psi(q^9) + 2\psi(q)^2\psi(q^3)\psi(q^9) \\
&= 5\psi(q)^2\psi(q^3)\psi(q^9) = 5\sum_{n=0}^{\infty} t'(1,1,3,9;n)q^n = \frac{5}{16}\sum_{n=0}^{\infty} t(1,1,3,9;n)q^n.
\end{aligned}$$

Comparing the coefficients of q^n we obtain

$$t(1,1,3,9;n) = \frac{2}{5}N(1,1,3,9;8n+14).$$

Now applying Lemma 2.2, (2.3) and (2.4) we deduce the result. □

3. Formulas for $t(1,1,4,4;n)$, $t(1,4,4,4;n)$, $t(1,2,2,4;n)$ and $t(1,2,4,4;n)$

Lemma 3.1. *Let $a, b, c, d, n \in \mathbb{N}$. Then*

$$t(a,b,c,d;n)$$

$$
\begin{aligned}
&= N(a,b,c,d;8n+a+b+c+d) - N(a,b,c,4d;8n+a+b+c+d)\\
&\quad - N(a,b,4c,d;8n+a+b+c+d) + N(a,b,4c,4d;8n+a+b+c+d)\\
&\quad - N(a,4b,c,d;8n+a+b+c+d) + N(a,4b,c,4d;8n+a+b+c+d)\\
&\quad + N(a,4b,4c,d;8n+a+b+c+d) - N(a,4b,4c,4d;8n+a+b+c+d)\\
&\quad - N(4a,b,c,d;8n+a+b+c+d) + N(4a,b,c,4d;8n+a+b+c+d)\\
&\quad + N(4a,b,4c,d;8n+a+b+c+d) - N(4a,b,4c,4d;8n+a+b+c+d)\\
&\quad + N(4a,4b,c,d;8n+a+b+c+d) - N(4a,4b,c,4d;8n+a+b+c+d)\\
&\quad - N(4a,4b,4c,d;8n+a+b+c+d) + N(4a,4b,4c,4d;8n+a+b+c+d).
\end{aligned}
$$

Proof. It is clear that

$$
\begin{aligned}
&t(a,b,c,d;n)\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = a\frac{x(x-1)}{2} + b\frac{y(y-1)}{2} + c\frac{z(z-1)}{2} + d\frac{w(w-1)}{2}\}\right|\\
&= \big|\{(x,y,z,w)\in\mathbb{Z}^4 \mid 8n+a+b+c+d\\
&\qquad\qquad = a(2x-1)^2 + b(2y-1)^2 + c(2z-1)^2 + d(2w-1)^2\}\big|\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+dw^2, 2 \mid xyzw-1\}\right|\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+dw^2, 2 \mid yzw-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+by^2+cz^2+dw^2, 2 \mid yzw-1\}\right|\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+dw^2, 2 \mid zw-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+cz^2+dw^2, 2 \mid zw-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+by^2+cz^2+dw^2, 2 \mid zw-1\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+4by^2+cz^2+dw^2, 2 \mid zw-1\}\right|\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+4cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+4cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+by^2+cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+by^2+4cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+4by^2+cz^2+dw^2, 2 \mid w-1\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+4by^2+4cz^2+dw^2, 2 \mid w-1\}\right|\\
&= \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+dw^2\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+cz^2+4dw^2\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+4cz^2+dw^2\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+by^2+4cz^2+4dw^2\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+cz^2+dw^2\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+cz^2+4dw^2\}\right|\\
&\quad + \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+4cz^2+dw^2\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = ax^2+4by^2+4cz^2+4dw^2\}\right|\\
&\quad - \left|\{(x,y,z,w)\in\mathbb{Z}^4 \mid n = 4ax^2+by^2+cz^2+dw^2\}\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+by^2+cz^2+4dw^2\}\big| \\
&+ \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+by^2+4cz^2+dw^2\}\big| \\
&- \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+by^2+4cz^2+4dw^2\}\big| \\
&+ \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+4by^2+cz^2+dw^2\}\big| \\
&- \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+4by^2+cz^2+4dw^2\}\big| \\
&- \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+4by^2+4cz^2+dw^2\}\big| \\
&+ \big|\{(x,y,z,w)\in\mathbb{Z}^4 \bigm| n = 4ax^2+4by^2+4cz^2+4dw^2\}\big|.
\end{aligned}
$$

Thus the result follows. □

For general positive integer n, in a series of papers A. Alaca, S. Alaca, M.F. Lemire and K.S. Williams (see [4,5,7]) gave explicit formulas for $N(a,b,c,d;n)$ in the cases $(a,b,c,d) = (1,1,4,4), (1,1,16,16), (1,1,4,16), (1,4,4,4), (1,4,16,16), (1,4,4,16), (1,2,2,4), (1,2,2,16), (1,2,16,16), (1,2,4,16), (1,2,4,8), (1,2,4,4), (1,2,8,16), (1,4,8,8)$ and $(1,8,8,16)$. Based on Lemma 3.1, we need some special results in [4,5,7] to prove our formulas for $t(1,1,4,4;n)$, $t(1,4,4,4;n)$, $t(1,2,2,4;n)$ and $t(1,2,4,4;n)$.

Lemma 3.2 ([4, Theorem 1.11]). *Let $n\in\mathbb{N}$ with $n\equiv 2 \pmod 4$. Then $N(1,1,4,4;n) = 4\sigma(n/2)$.*

Lemma 3.3 ([5, Theorems 4.6 and 4.8]). *Let $n\in\mathbb{N}$ and $n\equiv 2 \pmod 8$. Then*

$$N(1,1,16,16;n) = N(1,1,4,16;n) = 2\sigma\big(\frac{n}{2}\big) + 2\big(\frac{2}{n/2}\big)S_4\big(\frac{n}{2}\big).$$

Theorem 3.1. *Let $n\in\mathbb{N}$. Then*

$$t(1,1,4,4;n) = 2\big(\sigma(4n+5) + (-1)^n S_4(4n+5)\big).$$

Proof. Since $x^2 \not\equiv 2 \pmod 4$ for $x\in\mathbb{Z}$, from Lemma 3.1 we see that

$$
\begin{aligned}
&t(1,1,4,4;n) \\
&= N(1,1,4,4;8n+10) - N(1,1,4,16;8n+10) \\
&\quad - N(1,1,16,4;8n+10) + N(1,1,16,16;8n+10) - N(1,4,4,4;8n+10) \\
&\quad + N(1,4,4,16;8n+10) + N(1,4,16,4;8n+10) - N(1,4,16,16;8n+10) \\
&\quad - N(4,1,4,4;8n+10) + N(4,1,4,16;8n+10) + N(4,1,16,4;8n+10) \\
&\quad - N(4,1,16,16;8n+10) + N(4,4,4,4;8n+10) - N(4,4,4,16;8n+10) \\
&\quad - N(4,4,16,4;8n+10) + N(4,4,16,16;8n+10) \\
&= N(1,1,4,4;8n+10) - 2N(1,1,4,16;8n+10) + N(1,1,16,16;8n+10).
\end{aligned}
$$

Now applying Lemmas 3.2 and 3.3 we obtain

$$
\begin{aligned}
t(1,1,4,4;n) &= 4\sigma(4n+5) - 2\Big(2\sigma(4n+5) + 2\Big(\frac{2}{4n+5}\Big)S_4(4n+5)\Big) \\
&\quad + 2\sigma(4n+5) + 2\Big(\frac{2}{4n+5}\Big)S_4(4n+5) \\
&= 2\Big(\sigma(4n+5) - \Big(\frac{2}{4n+5}\Big)S_4(4n+5)\Big).
\end{aligned}
$$

This yields the result. □

Lemma 3.4 ([4, Theorem 1.18]). *Let $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$. Then*

$$N(1,4,4,4;n) = 2\sigma(n).$$

Lemma 3.5 ([5, Theorem 4.5]). *Let $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$. Then*

$$N(1,4,16,16;n) = \frac{1}{2}\sigma(n) + \frac{1}{2}(2 + (-1)^{\frac{n-1}{4}})S_4(n).$$

Lemma 3.6 ([5, Theorem 4.7]). *Let $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$. Then*

$$N(1,4,4,16;n) = \sigma(n) + S_4(n).$$

Theorem 3.2. *Let $n \in \mathbb{N}$. Then*

$$t(1,4,4,4;n) = \frac{1}{2}\Big(\sigma(8n+13) - 3S_4(8n+13)\Big).$$

Proof. Since $x^2 \equiv 0,1,4 \pmod 8$ for $x \in \mathbb{Z}$, using Lemma 3.1 we see that

$$\begin{aligned}
&t(1,4,4,4;n)\\
&= N(1,4,4,4;8n+13) - N(1,4,4,16;8n+13)\\
&\quad - N(1,4,16,4;8n+13) + N(1,4,16,16;8n+13) - N(1,16,4,4;8n+13)\\
&\quad + N(1,16,4,16;8n+13) + N(1,16,16,4;8n+13) - N(1,16,16,16;8n+13)\\
&\quad - N(4,4,4,4;8n+13) + N(4,4,4,16;8n+13) + N(4,4,16,4;8n+13)\\
&\quad - N(4,4,16,16;8n+13) + N(4,16,4,4;8n+13) - N(4,16,4,16;8n+13)\\
&\quad - N(4,16,16,4;8n+13) + N(4,16,16,16;8n+13)\\
&= N(1,4,4,4;8n+13) - 3N(1,4,4,16;8n+13) + 3N(1,4,16,16;8n+13).
\end{aligned}$$

Now applying Lemmas 3.4, 3.5 and 3.6 we obtain

$$\begin{aligned}
&t(1,4,4,4;n)\\
&= 2\sigma(8n+13) - 3(\sigma(8n+13) + S_4(8n+13)) + \frac{3}{2}(\sigma(8n+13) + S_4(8n+13))\\
&= \frac{1}{2}\Big(\sigma(8n+13) - 3S_4(8n+13)\Big).
\end{aligned}$$

This proves the theorem. □

Lemma 3.7 ([4, Theorem 1.14]). *Let $n \in \mathbb{N}$ with $2 \nmid n$. Then*

$$N(1,2,2,4;n) = 2\sigma(n).$$

Lemma 3.8 ([5, Theorems 4.9, 4.11 and 4.13]). *Let $n \in \mathbb{N}$ and $n \equiv 1 \pmod 8$. Then*

$$N(1,2,2,16;n) = N(1,8,8,16;n) = N(1,2,8,16;n) = \sigma(n) + S_4(n).$$

Lemma 3.9 ([5, Theorems 4.1 and 4.4]). *Let $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$. Then*

$$N(1,2,4,8;n) = N(1,4,8,8;n) = \sigma(n) + (-1)^{\frac{n-1}{4}}S_4(n).$$

Theorem 3.3. *Let $n \in \mathbb{N}$. Then*

$$t(1,2,2,4;n) = \sigma(8n+9) - S_4(8n+9).$$

Proof. From Lemma 3.1 we have

$$
\begin{aligned}
&t(1,2,2,4;n)\\
&= N(1,2,2,4;8n+9) - N(1,2,2,16;8n+9)\\
&\quad - N(1,2,4,8;8n+9) + N(1,2,8,16;8n+9) - N(1,2,4,8;8n+9)\\
&\quad + N(1,2,8,16;8n+9) + N(1,4,8,8;8n+9) - N(1,8,8,16;8n+9)\\
&\quad - N(4,2,2,4;8n+9) + N(4,2,2,16;8n+9) + N(4,2,8,4;8n+9)\\
&\quad - N(4,2,8,16;8n+9) + N(4,8,2,4;8n+9) - N(4,8,2,16;8n+9)\\
&\quad - N(4,8,8,4;8n+9) + N(4,8,8,16;8n+9)\\
&= N(1,2,2,4;8n+9) - N(1,2,2,16;8n+9) - 2N(1,2,4,8;8n+9)\\
&\quad + 2N(1,2,8,16;8n+9) + N(1,4,8,8;8n+9) - N(1,8,8,16;8n+9).
\end{aligned}
$$

Now applying Lemmas 3.7, 3.8 and 3.9 we obtain

$$
\begin{aligned}
&t(1,2,2,4;n)\\
&= 2\sigma(8n+9) - (\sigma(8n+9) + S_4(8n+9)) - 2(\sigma(8n+9) + S_4(8n+9))\\
&\quad + 2(\sigma(8n+9) + S_4(8n+9)) + \sigma(8n+9) + S_4(8n+9) - (\sigma(8n+9) + S_4(8n+9))\\
&= \sigma(8n+9) - S_4(8n+9),
\end{aligned}
$$

which completes the proof. □

Lemma 3.10 ([5, Theorems 4.17 and 4.18]). *Let $n \in \mathbb{N}$ and $n \equiv 1,3 \pmod 8$. Then*

$$N(1,2,4,16;n) = N(1,2,16,16;n) = S(n) + S_2(n).$$

Lemma 3.11 ([7, Theorem 5.4]). *Let $n \in \mathbb{N}$ with $2 \nmid n$. Then*

$$N(1,2,4,4;n) = 2S(n).$$

Theorem 3.4. *Let $n \in \mathbb{N}$. Then*

$$t(1,2,4,4;n) = -\sum_{d \mid 8n+11} d\left(\frac{2}{d}\right) - S_2(8n+11).$$

Proof. Since $x^2 \equiv 0,1 \pmod 4$ for $x \in \mathbb{Z}$, from Lemma 3.1 we see that

$$
\begin{aligned}
&t(1,2,4,4;n)\\
&= N(1,2,4,4;8n+11) - N(1,2,4,16;8n+11)\\
&\quad - N(1,2,16,4;8n+11) + N(1,2,16,16;8n+11) - N(1,8,4,4;8n+11)\\
&\quad + N(1,8,4,16;8n+11) + N(1,8,16,4;8n+11) - N(1,8,16,16;8n+11)\\
&\quad - N(4,2,4,4;8n+11) + N(4,2,4,16;8n+11) + N(4,2,16,4;8n+11)\\
&\quad - N(4,2,16,16;8n+11) + N(4,8,4,4;8n+11) - N(4,8,4,16;8n+11)\\
&\quad - N(4,8,16,4;8n+11) + N(4,8,16,16;8n+11)\\
&= N(1,2,4,4;8n+11) - 2N(1,2,4,16;8n+11) + N(1,2,16,16;8n+11).
\end{aligned}
$$

Now applying Lemmas 3.10 and 3.11 we obtain

$$t(1,2,4,4;n) = 2S(8n+11) - 2(S(8n+11) + S_2(8n+11)) + (S(8n+11) + S_2(8n+11))$$

$$= S(8n+11) - S_2(8n+11).$$

Since

$$S(8n+11) = \sum_{d\mid 8n+11} \frac{n}{d}\Big(\frac{2}{d}\Big) = \sum_{d\mid 8n+11} d\Big(\frac{2}{(8n+11)/d}\Big) = -\sum_{d\mid 8n+11} d\Big(\frac{2}{d}\Big),$$

from the above we deduce the result. □

4. Formulas for $t(1,3,3,9;n)$, $t(1,1,9,9;n)$, $t(1,9,9,9;n)$ and $t(1,1,1,9;n)$

For $a,b,c,d,n \in \mathbb{N}$ let

$$N_0(a,b,c,d;n) = \big|\big\{(x,y,z,w) \in \mathbb{Z}^4 \bigm| n = ax^2+by^2+cz^2+dw^2,\ 2\nmid xyzw\big\}\big|.$$

From the proof of Lemma 3.1 we know that

$$t(a,b,c,d;n) = N_0(a,b,c,d;8n+a+b+c+d). \tag{4.1}$$

Lemma 4.1. *Let $n \in \mathbb{N}$ and $n+1 = 2^\alpha 3^\beta n_1$ with $(6,n_1)=1$. Then*

$$t(1,1,3,3;n) = 2^{\alpha+4}\sigma(n_1).$$

Proof. By [8, Theorem 1.5],

$$t(1,1,3,3;n) = 16t'(1,1,3,3;n) = \begin{cases} 4N(1,1,3,3;n+1) & \text{if } 2\mid n,\\ 2(N(1,1,3,3;2n+2) - N(1,1,3,3;n+1)) & \text{if } 2\nmid n.\end{cases}$$

Ramanujan (see [9, pp. 114,223]) gave theta function identities equivalent to the following Liouville's conjecture (see [11]):

$$N(1,1,3,3;n+1) = \begin{cases} 4\sigma(n_1) & \text{if } 2\mid n,\\ 4(2^{\alpha+1}-3)\sigma(n_1) & \text{if } 2\nmid n.\end{cases}$$

Since $2n+2 = 2^{\alpha+1}3^\beta n_1$, combining all the above yields the result. □

Theorem 4.1. *Let $n \in \mathbb{N}$ and $n+2 = 2^\alpha 3^\beta n_1$ with $(6,n_1)=1$. Then*

$$t(1,3,3,9;n) = \begin{cases} 0 & \text{if } n \equiv 2 \pmod 3,\\ 2^{\alpha+4}\sigma(n_1) & \text{if } n\equiv 1 \pmod 3,\\ 2^{\alpha+3}\sigma(n_1) & \text{if } n \equiv 0 \pmod 3.\end{cases}$$

Proof. From (4.1) we know that $t(1,3,3,9;n) = N_0(1,3,3,9;8n+16)$. If $3\mid n-2$, then $8n+16 \equiv 2 \pmod 3$. Since $x^2 \not\equiv 2 \pmod 3$ for any $x \in \mathbb{Z}$, we get $t(1,3,3,9;n) = N_0(1,3,3,9;8n+16) = 0$. If $3 \mid n-1$, then $3 \mid 8n+16$ and so

$$\begin{aligned} t(1,3,3,9;n) &= N_0(1,3,3,9;8n+16)\\ &= \big|\big\{(x,y,z,w) \in \mathbb{Z}^4 \bigm| 8n+16 = (3x)^2+3y^2+3z^2+9w^2,\ 2\nmid xyzw\big\}\big| \end{aligned}$$

$$= \left|\left\{(x,y,z,w)\in\mathbb{Z}^4 \;\Big|\; \frac{8n+16}{3} = 3x^2+y^2+z^2+3w^2,\ 2\nmid xyzw\right\}\right|$$
$$= N_0(1,1,3,3;8(n-1)/3+8) = t(1,1,3,3;(n-1)/3).$$

If $3\mid n$, since $x^2+y^2\equiv 8n+16\equiv 1 \pmod 3$ implies $3\mid x$ or $3\mid y$ we see that

$$\begin{aligned}
t(1,1,3,3;n+1) &= N_0(1,1,3,3;8n+16)\\
&= \left|\left\{(x,y,z,w)\in\mathbb{Z}^4 \;\big|\; 8n+16=(3x)^2+y^2+3z^2+3w^2,\ 2\nmid xyzw\right\}\right|\\
&\quad + \left|\left\{(x,y,z,w)\in\mathbb{Z}^4 \;\big|\; 8n+16=x^2+(3y)^2+3z^2+3w^2,\ 2\nmid xyzw\right\}\right|\\
&= 2N_0(1,3,3,9;8n+16) = 2t(1,3,3,9;n).
\end{aligned}$$

Now combining the above with Lemma 4.1 yields the result. □

For variable q with $|q|<1$ define

$$q\prod_{n=1}^{\infty}(1-q^{6n})^4 = \sum_{n=1}^{\infty} c(n)q^n.$$

From [12, p.374] or [14, p.121] we know that

$$c(n) = \frac{1}{3}\sum_{\substack{x,y\in\mathbb{Z}\\ n=x^2+3xy+3y^2\\ 3\mid x-2,\ 2\mid y-1}} (-1)^x x.$$

Thus,

$$c(n) = \frac{1}{3}\sum_{\substack{x,y\in\mathbb{Z}\\ n=x^2+3x(1+2y)+3(1+2y)^2\\ x\equiv 2 \pmod 3}} (-1)^x x = \frac{1}{3}\sum_{\substack{x,y\in\mathbb{Z}\\ 4n=x^2+3(x+2+4y)^2\\ x\equiv 2 \pmod 3}} (-1)^x x$$

and so

(4.2) $$c(n) = \frac{1}{3}\sum_{\substack{4n=a^2+3b^2\ (a,b\in\mathbb{Z})\\ a\equiv 2 \pmod 3,\, b\equiv a+2 \pmod 4}} (-1)^a a.$$

Lemma 4.2 ([2, Theorems 1.5 and 1.6]). *For $n\in\mathbb{N}$ we have*

$$N(1,1,9,9;n) = \begin{cases} 4\sigma(n)-8\sigma(n/2) & \text{if } n\equiv 2,4 \pmod 6,\\ \frac{4}{3}\sigma(n) & \text{if } n\equiv 5 \pmod 6,\\ 8\sigma(n/9)-32\sigma(n/36) & \text{if } n\equiv 0 \pmod 6,\\ \frac{4}{3}\sigma(n)+\frac{8}{3}c(n) & \text{if } n\equiv 1 \pmod 6 \end{cases}$$

and

$$N(1,9,9,9;n) = \begin{cases} 8\sigma(n/9) & \text{if } n\equiv 3 \pmod 6,\\ 2\sigma(n)-4\sigma(n/2) & \text{if } n\equiv 4 \pmod 6,\\ 8\sigma(n/9)-32\sigma(n/36) & \text{if } n\equiv 0 \pmod 6. \end{cases}$$

Lemma 4.3 ([3, Theorems 2.5 and 2.10]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(1,1,36,36;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12} \end{cases}$$

and

$$N(1,4,36,36;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lemma 4.4 ([3, Theorem 2.4]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(1,1,9,36;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lemma 4.5 ([3, Theorem 2.8]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(1,4,9,9;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lemma 4.6 ([3, Theorem 2.9]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(1,4,9,36;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lemma 4.7 ([3, Theorem 2.15]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(4,4,9,9;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Lemma 4.8 ([3, Theorem 2.16]). *For $n \in \mathbb{N}$ with $4 \mid n$ we have*

$$N(4,4,9,36;n) = \begin{cases} \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) + \frac{8}{3}c(n/4) & \text{if } n \equiv 4 \pmod{12}, \\ \frac{4}{3}\sigma(n/4) - \frac{16}{3}\sigma(n/16) & \text{if } n \equiv 8 \pmod{12}, \\ 8\sigma(n/36) - 32\sigma(n/144) & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Theorem 4.2. *Let $n \in \mathbb{N}$. Then*

$$t(1,1,9,9;n) = \begin{cases} \frac{8}{3}\sigma(2n+5) & \text{if } n \equiv 0 \pmod 3, \\ 16\sigma(\frac{2n+5}{9}) & \text{if } n \equiv 2 \pmod 9, \\ 0 & \text{if } n \equiv 5,8 \pmod 9, \\ \frac{8}{3}(\sigma(2n+5) - c(2n+5)) & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proof. For $n \equiv 2 \pmod 3$ we see that $3 \mid 8n+20$ and so

$$\begin{aligned} & t(1,1,9,9;n) \\ &= N_0(1,1,9,9;8n+20) \\ &= \left|\left\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+20 = (3x)^2 + (3y)^2 + 9z^2 + 9w^2,\ 2 \nmid xyzw\right\}\right| \\ &= \begin{cases} 0 & \text{if } 9 \nmid n-2, \\ N_0(1,1,1,1;\frac{8n+20}{9}) = t(1,1,1,1;\frac{n-2}{9}) = 16\sigma(\frac{2n+5}{9}) & \text{if } 9 \mid n-2. \end{cases} \end{aligned}$$

Now assume $n \equiv 0,1 \pmod 3$. By Lemma 3.1,

$$\begin{aligned} & t(1,1,9,9;n) \\ &= N(1,1,9,9;8n+20) - 2N(1,1,9,36;8n+20) + N(1,1,36,36;8n+20) \\ &\quad - 2N(1,4,9,9;8n+20) + 4N(1,4,9,36;8n+20) - 2N(1,4,36,36;8n+20) \\ &\quad + N(4,4,9,9;8n+20) - 2N(4,4,9,36;8n+20) + N(1,1,9,9;2n+5). \end{aligned}$$

For $n \equiv 0 \pmod 3$ applying Lemmas 4.2-4.8 we see that

$$\begin{aligned} & t(1,1,9,9;n) \\ &= 4\sigma(8n+20) - 8\sigma(\frac{8n+20}{2}) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16})) \\ &\quad + \frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16})) \\ &\quad + 4(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16})) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16})) \\ &\quad + \frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16})) + \frac{4}{3}\sigma(2n+5) \\ &= 28\sigma(2n+5) - 24\sigma(2n+5) - \frac{8}{3}\sigma(2n+5) + \frac{32}{3}\sigma(\frac{4n+5}{4}) + \frac{4}{3}\sigma(2n+5) \\ &= \frac{8}{3}\sigma(2n+5). \end{aligned}$$

For $n \equiv 1 \pmod 3$, applying Lemmas 4.2-4.8 we find that

$$\begin{aligned} & t(1,1,9,9;n) \\ &= 4\sigma(8n+20) - 8\sigma(\frac{8n+20}{2}) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) + \frac{8}{3}c(\frac{8n+20}{4})) \\ &\quad + \frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) + \frac{8}{3}c(\frac{8n+20}{4}) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) \\ &\quad + \frac{8}{3}c(\frac{8n+20}{4})) + 4(\frac{4}{3}\sigma(\frac{8n+20}{4}) - \frac{16}{3}\sigma(\frac{8n+20}{16}) + \frac{8}{3}c(\frac{8n+20}{4})) - 2(\frac{4}{3}\sigma(\frac{8n+20}{4}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{16}{3}\sigma(\frac{8n+20}{16})+\frac{8}{3}c(\frac{8n+20}{4}))+\frac{4}{3}\sigma(\frac{8n+20}{4})-\frac{16}{3}\sigma(\frac{8n+20}{16})+\frac{8}{3}c(\frac{8n+20}{4})\\
&-2(\frac{4}{3}\sigma(\frac{8n+20}{4})-\frac{16}{3}\sigma(\frac{8n+20}{16})+\frac{8}{3}c(\frac{8n+20}{4}))+\frac{4}{3}\sigma(2n+5)+\frac{8}{3}c(2n+5)\\
=\ &28\sigma(2n+5)-24\sigma(2n+5)-\frac{8}{3}\sigma(2n+5)-\frac{16}{3}c(2n+5)+\frac{4}{3}\sigma(2n+5)+\frac{8}{3}c(2n+5)\\
=\ &\frac{8}{3}(\sigma(2n+5)-c(2n+5)).
\end{aligned}
$$

The proof is now complete. □

Theorem 4.3. *Let $n\in\mathbb{N}$. Then*

$$
t(1,9,9,9;n)=\begin{cases}16\sigma(\frac{2n+7}{9}) & \text{if } n\equiv 1 \pmod 9,\\ 0 & \text{if } n\equiv 2,4,5,7,8 \pmod 9,\\ \frac{4}{3}(\sigma(2n+7)-c(2n+7)) & \text{if } n\equiv 0 \pmod 3.\end{cases}
$$

Proof. For $x\in\mathbb{Z}$ we see that $x(x-1)/2\equiv 0,1,3,6 \pmod 9$. Thus, $t(1,9,9,9;n)=0$ for $n\equiv 2,4,5,7,8 \pmod 9$. Now we assume that $n\equiv 0,1,3,6 \pmod 9$. For $n\equiv 1 \pmod 9$ we see that $9\mid 8n+28$ and so

$$
\begin{aligned}
t(1,9,9,9;n)&=N_0(1,9,9,9;8n+28)\\
&=\big|\{(x,y,z,w)\in\mathbb{Z}^4\ \big|\ 8n+28=(3x)^2+9y^2+9z^2+9w^2,\ 2\nmid xyzw\}\big|\\
&=N_0\big(1,1,1,1;\frac{8n+28}{9}\big)=t\big(1,1,1,1;\frac{n-1}{9}\big)=16\sigma\big(\frac{2n+7}{9}\big).
\end{aligned}
$$

For $n\equiv 0 \pmod 3$ we see that

$$
\begin{aligned}
t(1,1,9,9;n+1)&=N_0(1,1,9,9;8n+28)\\
&=\big|\{(x,y,z,w)\in\mathbb{Z}^4\ \big|\ 8n+28=(3x)^2+y^2+9z^2+9w^2,\ 2\nmid xyzw\}\big|\\
&\quad+\big|\{(x,y,z,w)\in\mathbb{Z}^4\ \big|\ 8n+28=x^2+(3y)^2+9z^2+9w^2,\ 2\nmid xyzw\}\big|\\
&=2N_0(1,9,9,9;8n+28)=2t(1,9,9,9;n).
\end{aligned}
$$

Now applying Theorem 4.2 we deduce the result in this case. □

Theorem 4.4. *Let $n\in\mathbb{N}$. Then*

$$
t(1,1,1,9;n)=\begin{cases}4\sigma(2n+3)+12\sigma(\frac{2n+3}{9}) & \text{if } n\equiv 0 \pmod 3,\\ 8\sigma(2n+3) & \text{if } n\equiv 1 \pmod 3,\\ 4(\sigma(2n+3)-c(2n+3)) & \text{if } n\equiv 2 \pmod 3.\end{cases}
$$

Proof. For $n\equiv 0 \pmod 3$ we see that $3\mid 8n+12$. If $8n+12=x^2+y^2+z^2+w^2$ for $x,y,z,w\in\mathbb{Z}$, then either $x\equiv y\equiv z\equiv w\equiv 0 \pmod 3$ or $xyzw\equiv\pm 3 \pmod 9$. Thus,

$$
N_0(1,1,1,1;8n+12)=4N_0(1,1,1,9;8n+12)-3N_0(9,9,9,9;8n+12).
$$

This together with (4.1) yields

$$
t(1,1,1,1;n+1)=\begin{cases}4t(1,1,1,9;n) & \text{if } n\equiv 0,6 \pmod 9,\\ 4t(1,1,1,9;n)-3t(1,1,1,1;\frac{n-3}{9}) & \text{if } n\equiv 0 \pmod 3.\end{cases}
$$

Now combining the above with (1.2) yields the result in this case.

Suppose $n \equiv 1 \pmod 3$. Then $8n+12 \equiv 2 \pmod 3$. If $8n+12 = x^2+y^2+z^2+9w^2$ for $x,y,z,w \in \mathbb{Z}$, then $3 \mid xyz$ but $9 \nmid xyz$. Thus,

$$\begin{aligned} t(1,1,1,9;n) &= N_0(1,1,1,9;8n+12) \\ &= \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = (3x)^2+y^2+z^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &\quad + \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = x^2+(3y)^2+z^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &\quad + \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = x^2+y^2+(3z)^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &= 3N_0(1,1,9,9;8n+12) = 3t(1,1,9,9;n-1). \end{aligned}$$

This together with Theorem 4.2 yields the result in this case.

For $n \equiv 2 \pmod 3$ we see that $8n+12 \equiv 1 \pmod 3$ and so

$$\begin{aligned} t(1,1,1,9;n) &= N_0(1,1,1,9;8n+12) \\ &= \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = (3x)^2+(3y)^2+z^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &\quad + \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = x^2+(3y)^2+(3z)^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &\quad + \big|\{(x,y,z,w) \in \mathbb{Z}^4 \mid 8n+12 = (3x)^2+y^2+(3z)^2+9w^2,\ 2 \nmid xyzw\}\big| \\ &= 3N_0(1,9,9,9;8n+12) = 3t(1,9,9,9;n-2). \end{aligned}$$

Now combining the above with Theorem 4.3 yields the result in the case $n \equiv 2 \pmod 3$. The proof is now complete. □

In conclusion we pose the following conjecture.

Conjecture 4.1. *Suppose $n \in \mathbb{N}$ and $8n+9 = 3^\beta n_1$ with $3 \nmid n_1$. Then*

$$t(1,1,3,4;n) = \frac{1}{2}\Big(3^{\beta+1}\Big(\frac{3}{n_1}\Big)-1\Big)\sum_{d \mid n_1} d\Big(\frac{3}{d}\Big) - \sum_{\substack{a,b \in \mathbb{N},\ 2 \nmid a \\ 4(8n+9)=a^2+3b^2}} (-1)^{\frac{a-1}{2}} a.$$

Conjecture 4.1 has been checked for $n \le 1000$.

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